M100 27

THE OPEN UNIVERSITY

Mathematics Foundation Course Unit 27

Complex Numbers I

The Open University

Mathematics Foundation Course Unit 27

COMPLEX NUMBERS I

Prepared by the Mathematics Foundation Course Team

Correspondence Text 27

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Contents

Objectives

The principal aim of this unit is to introduce complex numbers and some of their properties.

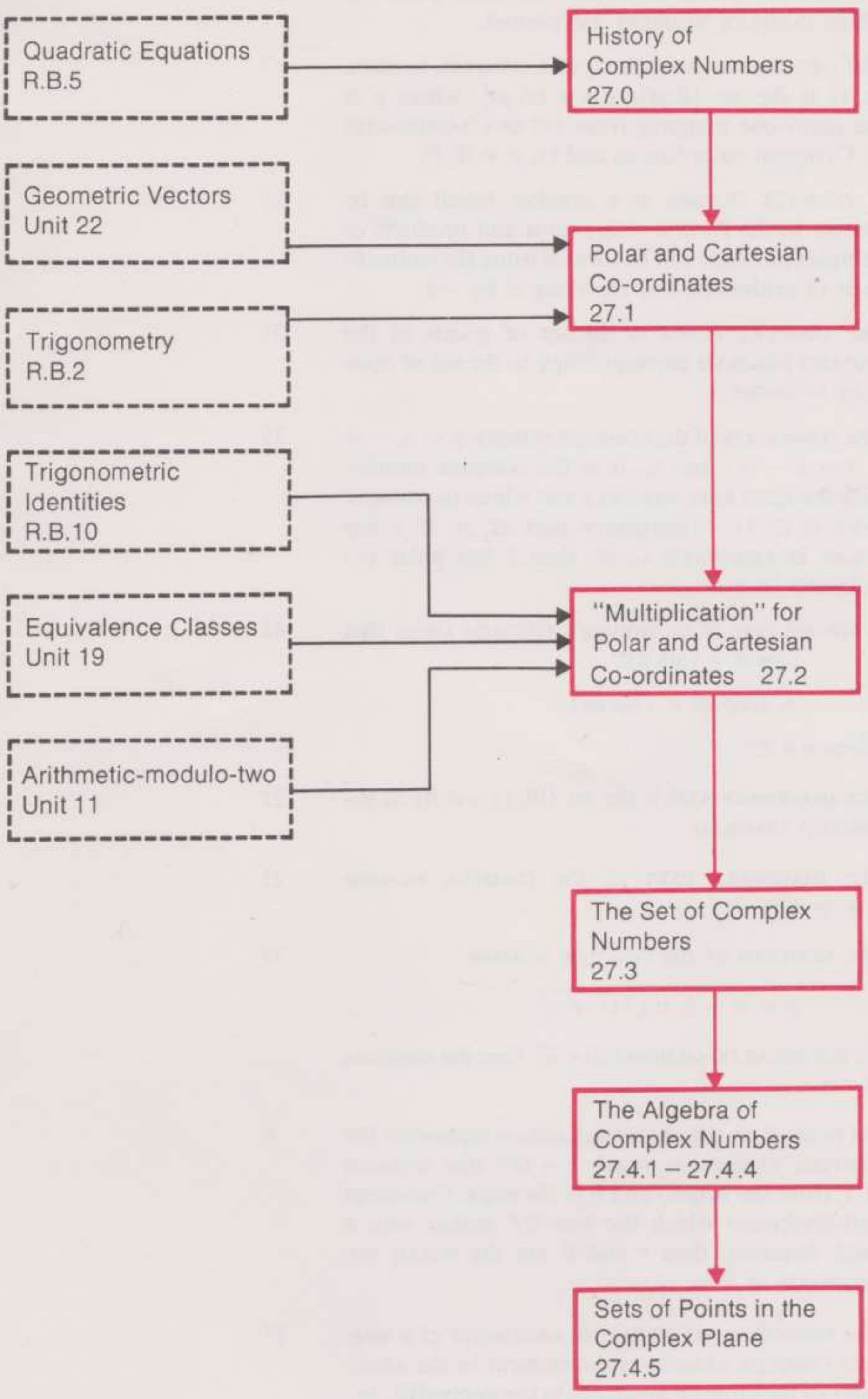
After working through this unit you should be able to:

(i) use a one-one function to transfer polar co-ordinates to Cartesian co-ordinates and vice versa;
(ii) multiply complex numbers given in either polar or Cartesian form, and appreciate the relationship between the corresponding binary operations for the two forms;
(iii) find the argument and the principal value of the argument of a complex number and of the product of two complex numbers (for non-zero complex numbers);
(iv) manipulate complex numbers using the $x + iy$ notation;
(v) find the real and imaginary parts of an expression involving complex numbers;
(vi) find the roots of any quadratic equation;
(vii) plot complex numbers on an Argand diagram;
(viii) divide complex numbers given in either polar or Cartesian form;
(ix) find the modulus and the conjugate of a complex number;
(x) use De Moivre's theorem to find integer powers of complex numbers;
(xi) recognize circles, straight lines, and regions bounded by circles or straight lines, specified in terms of complex numbers.

Note

Before working through this correspondence text, make sure you have read the general introduction to the mathematics course in the Study Guide, as this explains the philosophy underlying the whole course. You should also be familiar with the section which explains how a text is constructed and the meanings attached to the stars and other symbols in the margin, as this will help you to find your way through the text.

Structural Diagram



Glossary

Page

Terms which are defined in this glossary are printed in CAPITALS.

AMPLITUDE	See ARGUMENT.	
ARGAND DIAGRAM	An ARGAND DIAGRAM is the Cartesian plane on which COMPLEX NUMBERS are plotted.	21
ARGUMENT	The ARGUMENT (AMPLITUDE) of a COMPLEX NUMBER (x, y) is the set $\{\theta : p((r, \theta)) = (x, y)\}$, where p is the many-one mapping from POLAR CO-ORDINATES to Cartesian co-ordinates and $(x, y) \neq (0, 0)$.	17
COMPLEX NUMBER	A COMPLEX NUMBER is a number which can be written in the form $x + iy$; sums and products of complex numbers can be formed using the ordinary rules of arithmetic and replacing i^2 by -1.	21
COMPLEX PLANE	The COMPLEX PLANE is the set of points in the ARGAND DIAGRAM corresponding to the set of COMPLEX NUMBERS.	21
CONJUGATE	The CONJUGATE of the COMPLEX NUMBER $z = x + iy$ is $\bar{z} = x - iy$; that is, it is the complex number with the same REAL PART as z and whose IMAGINARY PART is $(-1) \times$ (imaginary part of z). If z has POLAR CO-ORDINATES (r, θ), then $\bar{z}$ has polar co-ordinates $(r, -\theta)$.	35
DE MOIVRE'S THEOREM	A special case of DE MOIVRE'S THEOREM states that $(\cos \theta + i \sin \theta)^n = (\cos n\theta + i \sin n\theta)$, where $n \in Z^+$.	42
IMAGINARY AXIS	The IMAGINARY AXIS is the set $\{(0, y) : y \in R\}$ in the ARGAND DIAGRAM.	21
IMAGINARY PART	The IMAGINARY PART of the COMPLEX NUMBER $x + iy$ is y.	21
MODULUS	The MODULUS of the COMPLEX NUMBER $z = x + iy$ is $\sqrt{x^2 + y^2}$. If z has POLAR CO-ORDINATES (r, θ), then the modulus of z is r.	35
POLAR CO-ORDINATES	If a point P on the ARGAND DIAGRAM represents the COMPLEX NUMBER z, where $r = OP$ (the distance of P from the origin) and θ is the angle (measured anti-clockwise) which the line OP makes with a fixed direction, then r and θ are the POLAR CO-ORDINATES of P (or z).	6
PRINCIPAL VALUE OF THE ARGUMENT	The PRINCIPAL VALUE OF THE ARGUMENT of a non-zero COMPLEX NUMBER is the element in the ARGUMENT of the number which lies in the interval $[0, 2\pi[$.	17
REAL AXIS	The REAL AXIS is the set $\{(x, 0) : x \in R\}$ in the ARGAND DIAGRAM.	21
REAL PART	The REAL PART of the COMPLEX NUMBER $x + iy$ is x.	21
TRIANGLE INEQUALITY	The TRIANGLE INEQUALITY for COMPLEX NUMBERS is $\lvert z_1 + z_2 \rvert \leqslant \lvert z_1 \rvert + \lvert z_2 \rvert$.	39

Notation

The symbols are presented in the order in which they appear in the text.

		Page
$\mathscr{R}\underset{\sim}{a}$	The geometric vector obtained by rotating the geometric vector $\underset{\sim}{a}$ about its blunt end-point through 90° anti-clockwise.	4
(r, θ)	The polar co-ordinates of a point (or a complex number).	5
R_0^+	The set of positive real numbers and zero.	7
p	The many-one mapping $(r,\theta) \longmapsto (r\cos\theta, r\sin\theta)$, $((r,\theta) \in R_0^+ \times R)$.	7
A	The set $\{(r,\theta): r \in R^+, 0 \leqslant \theta < 2\pi\} \cup \{(0,0)\}$.	7
P	The one-one mapping $(r,\theta) \longmapsto (r\cos\theta, r\sin\theta)$, $((r,\theta) \in A)$.	8
$\oplus_{2\pi}$	Addition modulo 2π.	13
$\otimes$	The operation on the set of Cartesian co-ordinates which corresponds to multiplication on the set of polar co-ordinates.	14
$\arg(x, y)$	The argument of (x, y).	17
$\operatorname{Arg}(x, y)$	The principal value of the argument of (x, y).	17
$x + iy$, z	The complex number (x, y).	23, 24
$\operatorname{Re} z$	The real part of z.	24
$\operatorname{Im} z$	The imaginary part of z.	24
$\lvert z\rvert$	The modulus of z.	35
$\bar{z}$	The conjugate of z.	35
C_1	The set of non-zero complex numbers.	37

Bibliography

A light, and very brief, introduction to complex numbers can be found in the paperback:

W. W. Sawyer, *Mathematician's Delight* (Penguin Books, 1943)

For a fuller introduction to complex numbers and their properties, see

F. J. Budden, *Complex Numbers and Their Applications* (Longmans, 1968).

27.0 INTRODUCTION

27.0

Introduction

In order to understand fully what complex numbers are, and why they are important, we need to know a little of their history. The complex number system is a natural generalization of the real number system, and this generalization was anticipated by the early Greek mathematicians. The basic question which faced the ancients was this:

Is there a number which when multiplied by itself gives -1?

It was not difficult for them to decide that there was no such number, for they argued, quite rightly, that the square of a positive or negative quantity must always be positive. On the other hand, it was disconcerting for them to have equations which had solutions only if one allowed the existence of $\sqrt{-1}$.

Diophantus (*c.*275 A.D.) was one of the first mathematicians to recognize that the set of real numbers is, in a sense, incomplete. He attempted to solve the apparently reasonable problem of finding the sides of a right-angled triangle of perimeter 12 and area 7. This leads directly to the equation (in modern notation)

$$6x^2 - 43x + 84 = 0,$$

in which x is the length of one side of the triangle.

This equation has roots which involve the square root of a negative quantity.

(See RB5)

The ancient mathematicians interpreted an equation of this kind as representing an impossible occurrence. Pacioli (1494) stated that the equation $x^2 + c = bx$ *cannot* be solved unless $b^2 \geqslant 4c$, and Cardan (1545) described the equation $x^4 + 12 = 6x^2$ as being "impossible", referring to the roots of such equations as "fictitious". However, Cardan did use the square root of a negative number in computation in order to divide 10 into two parts whose product is 40, and he found the two parts to be $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$. Gauss first called expressions of this kind "complex numbers".

Girolamo Cardano

In these early days complex numbers had a certain mystical quality. Mathematicians were sure that they did not exist; and yet, if one supposed that they did, then it was possible to solve certain problems very quickly. They were used as a calculating device, but regarded with deep suspicion, and that suspicion is still reflected in the words we use today. We still talk of a "real" number and an "imaginary" number, as if one were more "real" than the other.

The major misunderstanding of all the early mathematicians was that they had not appreciated that mathematics, unlike physics, is not something which exists, waiting for men to discover its intricacies, but it is man's own creation. Thus, "the square root of minus one" exists *if we say that it exists*; it is up to us to attach meaning to the phrase, which should, of course, be consistent with any previous definitions which we wish to include in the system under consideration.

Karl Friedrich Gauss

Wallis (1673) seems to have appreciated the point. He stated that the square root of a negative number was thought to imply the impossible, but that the same might also be said of a negative number, although we can easily explain the latter in a physical application:

> "These *Imaginary* Quantities (as they are commonly called), arising from the *Supposed* Root of a Negative Square (when they happen,), are reputed to imply that the Case proposed is Impossible.
>
> And so indeed it is, as to the first and strict notion of what is proposed. For it is not possible that any Number (Negative or Affirmative) Multiplied into itself can produce (for instance) -4. Since that Like Signs (whether $+$ or $-$) will produce $+$; and therefore not -4.

But it is also Impossible that any Quantity (though not a Supposed Square) can be *Negative*. Since that it is not possible that any *Magnitude* can be *Less than Nothing* or any *Number Fewer than None*.

Yet is not that Supposition (of Negative Quantities,) either Unuseful or Absurd; when rightly understood. And though, as to the bare Algebraick Notation, it import a Quantity less than nothing: Yet, when it comes to a Physical Application, it denotes as Real a Quantity as if the Sign were $+$; but to be interpreted in a contrary sense."

John Wallis

In this unit we shall re-examine the problem of defining $\sqrt{-1}$ in the light of our knowledge of sets, mappings and functions.

27.1 A NEW "SQUARE" FUNCTION

27.1

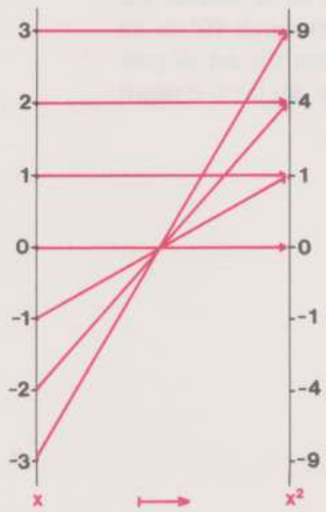
27.1.0 Introduction

27.1.0

Introduction

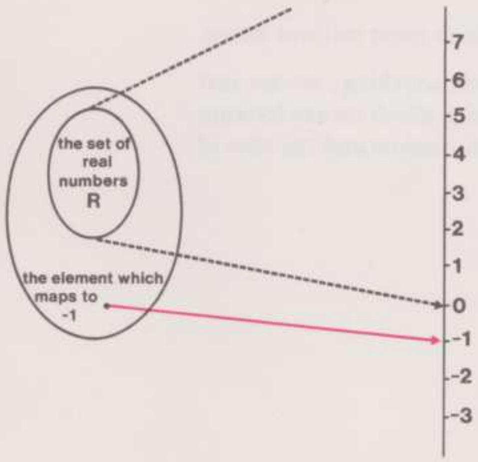
Since we are interested in defining $\sqrt{-1}$, we shall begin by looking at the "square" function:

$$f: x \longmapsto x^2 \qquad (x \in R).$$



Notice that R is a suitable codomain for f, since it contains all the images of R under f, although *all* the images are in fact greater than or equal to zero. This is simply another way of stating what the ancients knew: the square of any number is always positive (or zero). The number -1 is certainly not an image of any element in the domain of f.

We appear to be no further forward, but perhaps it is our definition of the "square" function which is unsatisfactory? Can we enlarge the domain of f to include an element which maps to -1? Can we define a new, more satisfactory "square" function?

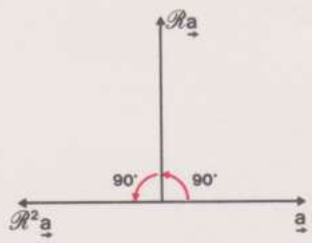


Setting aside for the moment the initial problem of how to specify this larger domain, there is going to be one major problem. In order to "square" something we must be able to "multiply" it by itself. So we are going to need a definition of "multiplication" on this new domain.

This then is our line of attack. We shall specify a new set, and introduce an operation on that set which we shall call "multiplication". Then we can easily define a "square" function in terms of this operation. Having done that, we can look for the element which maps to -1 under this function. This approach will give us a firm foundation for the study of *complex numbers.*

Which set should we choose for the domain of our new "square" function? The following idea will give us the clue.

Suppose that we start with a geometric vector $\underset{\sim}{a}$ and let $\mathscr{R}$ denote the mapping which "rotates $\underset{\sim}{a}$ about its blunt end-point through 90° in an anti-clockwise direction". This gives us a mapping from the set of geometric vectors to the set of geometric vectors. We write $\mathscr{R}\underset{\sim}{a}$ for the result of rotating $\underset{\sim}{a}$ through 90° anti-clockwise.



The point to notice is this: applying $\mathscr{R}$ twice to $\underset{\sim}{a}$ gives $\mathscr{R} \circ \mathscr{R}\underset{\sim}{a}$ which is equal to $-\underset{\sim}{a}$. In other words, if we write $\mathscr{R}^2$ for $\mathscr{R}$ applied twice, then

$$\mathscr{R}^2\underset{\sim}{a} = -\underset{\sim}{a} = -1 \times \underset{\sim}{a}.$$

We have

$$\mathscr{R}^2 = -I,$$

where I is the *identity mapping*:

$$I : \underset{\sim}{a} \longmapsto \underset{\sim}{a}.$$

There are, conceivably, many other sets and mappings on those sets, for which

$$(mapping)^2\,(element) = -(element).$$

We intend to choose the set and the mapping which most suit our needs.

It is important to realize that we have not proved anything; we are just led to an intuitive idea that the set and the mapping which we are looking for may well have something to do with geometric vectors and the idea of rotation.

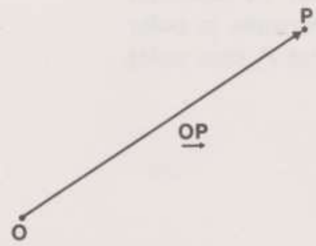
27.1.1 The Set of Geometric Vectors

27.1.1

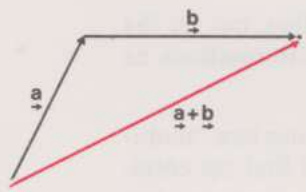
Discussion

* *

We saw in *Unit 22, Linear Algebra I* that once we specify a fixed point, which we call the origin, then all the points in two or three dimensions can be specified by geometric vectors. If O denotes the origin, then the geometric vector $\underrightarrow{OP}$ determines the point P.



We are concerned here with the set of geometric vectors lying in a *plane*; we know that such a set forms a vector space of dimension two, with suitable definitions of multiplication by a scalar and addition (illustrated in the following diagram). We shall call this set V.



Previously we constructed an interesting example of an "algebra" by introducing a further operation, the inner product. On this occasion we adopt an alternative notion of "multiplication" based on the idea of rotation about O. It turns out that this new "multiplication" leads us to a very satisfactory algebra with almost all the desirable properties of the algebra of real numbers. Before introducing this new "multiplication" on the set of geometric vectors, we shall need to consider the problem of notation; we do this in the next section.

27.1.2 Polar and Cartesian Co-ordinates

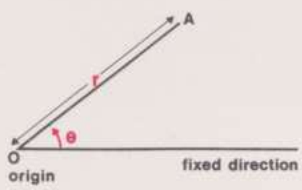
27.1.2

Main Text

* * *

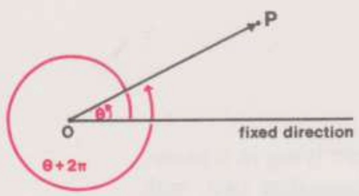
Geometric vectors (and also, of course, the points in a plane) can be represented either by *polar co-ordinates* (r, θ) or by *Cartesian co-ordinates* (x, y). (We shall use red brackets and black brackets to distinguish the two meanings of the number pairs in this correspondence text. This distinction is not usually made in books but it may be helpful to you initially.)

In case you haven't met polar co-ordinates we shall describe them briefly. To obtain polar co-ordinates, we choose a fixed point, called the origin, and a fixed direction.



Then given any general point A in the plane, we can specify its position by the angle θ (measured positive in an anti-clockwise direction from the fixed line) and the distance r of A from the origin. Then the numbers r and θ are called polar co-ordinates of A, or of the geometric vector $\underset{\longrightarrow}{OA}$. The origin O has polar co-ordinates $(0, \theta)$, where θ is arbitrary. **Definition 1**

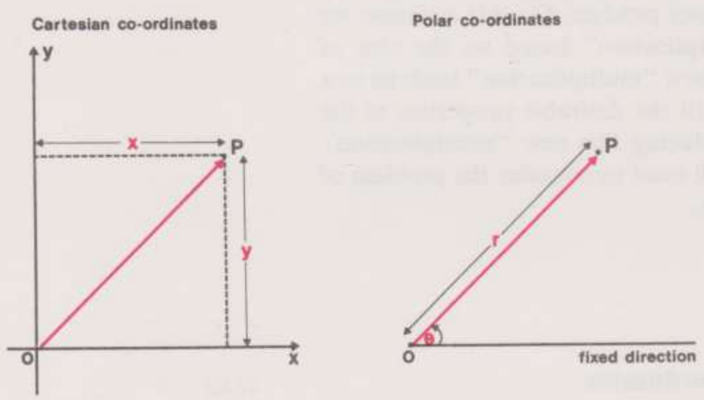
Polar co-ordinates are ideally suited to problems involving rotations and so they seem an obvious choice here. However, polar co-ordinates suffer from two disadvantages. Vector addition is cumbersome in polar co-ordinates, and, if we are given the origin O and a point P, then polar co-ordinates of P are not determined uniquely.



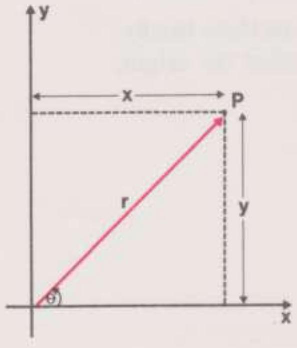
If θ gives the direction of $\underset{\longrightarrow}{OP}$ in radians, then so also do the angles $\theta + 2k\pi, k = \pm 1, \pm 2, \ldots$

Cartesian co-ordinates do not suffer from these difficulties, but on the other hand they are rather cumbersome when dealing with rotations, as we shall see.

We shall try to get the best from both systems by defining our new "multiplication" in terms of polar co-ordinates; then we shall find the corresponding operation in terms of Cartesian co-ordinates. First we must consider carefully the relationship between the two co-ordinate systems.



If we are given the polar co-ordinates of a point, can we determine its Cartesian co-ordinates, and vice versa?



In terms of mappings: is there a function which maps polar co-ordinates to Cartesian co-ordinates?

If we take the same origin in both cases, and the fixed direction for our polar co-ordinates along the positive x-axis, then clearly

$$x = r\cos\theta,$$

and

$$y = r\sin\theta.$$

So

$$\underset{\text{polar}}{p:(r,\theta)} \longmapsto \underset{\text{Cartesian}}{(r\cos\theta, r\sin\theta)} \qquad ((r,\theta)\in R_0^+ \times R)$$

is the required mapping, and it is indeed a function. (R_0^+ denotes the set of positive real numbers and zero.)

Notice that

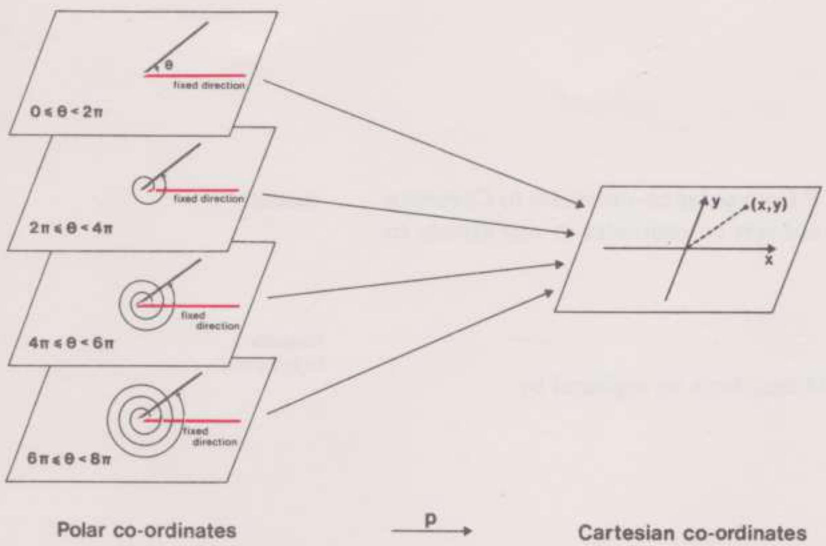
$$(r, \theta + 2\pi) \longmapsto (r\cos\theta, r\sin\theta)$$

and in general that

$$(r, \theta + 2k\pi) \longmapsto (r\cos\theta, r\sin\theta) \qquad (k = 0, \pm 1, \pm 2, \ldots).$$

Also

$$(0,\theta) \longmapsto (0,0) \qquad (\theta \in R).$$



The function p is many-one

The mapping p is thus many-one, so its reverse is one-many and is therefore *not* a function. This is highly undesirable, since it means that we cannot get back from Cartesian co-ordinates to polar co-ordinates uniquely. But by restricting the domain of p the situation can be improved.

Instead of taking $R_0^+ \times R$ as the domain, we take the set

$$A = \{(r,\theta): r\in R^+, 0 \leqslant \theta < 2\pi\} \cup \{(0,0)\}.$$

On this restricted domain, p is now one-one. We have cut out all the $(0,\theta)$'s except $(0,0)$ and we have cut out all the $\theta + 2k\pi$'s except θ itself.

Of course, changing the domain of p changes p: the domain is an integral part of the function. We shall use a capital P for this new function:

$$P:(r,\theta)\longmapsto(r\cos\theta, r\sin\theta) \qquad ((r,\theta)\in R^+ \times [0,2\pi[).$$

$$P:(0,0)\longmapsto(0,0).$$

Polar co-ordinates

Cartesian co-ordinates

The function P is one-one

The reverse mapping, P^{-1}, is now also a function.

If we are given (x, y), then our problem is to find the corresponding (r, θ) in A. The value of r is easily determined, because from Pythagoras' theorem

$$r = \sqrt{x^2 + y^2}.$$

If $r = 0$, then the polar co-ordinates are $(0, 0)$; if $r \neq 0$, the angle θ is determined uniquely once we calculate

(a) $\sin\theta = \dfrac{y}{r}$ (See RB2)

and

(b) $\cos\theta = \dfrac{x}{r}$

since we know that $0 \leqslant \theta < 2\pi$.

Summary

We have found a one-one function P from polar co-ordinates to Cartesian co-ordinates, which enables us to convert co-ordinates in one system to those in the other system. **Summary** * *

Exercise 1 **Exercise 1** (4 minutes)

Do we need both (a) and (b)? Could they both be replaced by

$$\tan\theta = \frac{y}{x}$$

when $x \neq 0$?

(i) Find two angles θ, such that $0 \leqslant \theta < 2\pi$, for which (See RB2)

$$\sin\theta = \tfrac{1}{2}.$$

(ii) Find two angles θ, such that $0 \leqslant \theta < 2\pi$, for which

$$\cos\theta = \frac{\sqrt{3}}{2}.$$

(iii) Write down the single angle θ, such that $0 \leqslant \theta < 2\pi$, for which

$$\sin\theta = \frac{1}{2} \textit{ and } \cos\theta = \frac{\sqrt{3}}{2}.$$

(iv) Find the two angles θ, such that $0 \leqslant \theta < 2\pi$, for which

$$\tan\theta = \frac{1}{2} \times \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}}. \quad ■$$

Exercise 2

Exercise 2
(2 minutes)

Find the images under p of the following

(i) $\left(1, \frac{\pi}{4}\right)$

(ii) $\left(2, \frac{\pi}{3}\right)$

What are the images of these elements under P? ■

Exercise 3

Exercise 3
(3 minutes)

Find the images of the following under the reverse of p.

(i) $(1, \sqrt{3})$

(ii) $(\sqrt{2}, -\sqrt{2})$

(iii) $(0, 0)$

(iv) $(0, 1)$

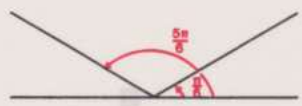
What are the images of these elements under the reverse of P? ■

Solution 1

Solution 1

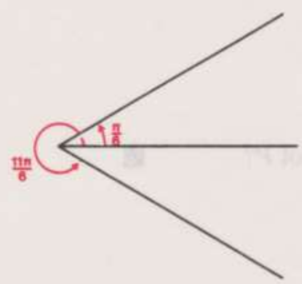
We do indeed require both (a) and (b) and the rest of the exercise is intended to show why.

(i)



Answers: $\dfrac{\pi}{6}$ and $\dfrac{5\pi}{6}$

(ii)



Answers: $\dfrac{\pi}{6}$ and $\dfrac{11\pi}{6}$

(iii) $\dfrac{\pi}{6}$

(iv) $\dfrac{\pi}{6}$ and $\dfrac{7\pi}{6}$

In general, any *one* of the equations

$$\sin \theta = a$$

$$\cos \theta = b$$

$$\tan \theta = c$$

where $-1 < a < 1$, $-1 < b < 1$, will have *two* solutions in the interval $[0, 2\pi[$ corresponding to *two* points in the plane. For instance, the points with Cartesian co-ordinates (x, y) and $(-x, -y)$ both have a polar co-ordinate θ satisfying

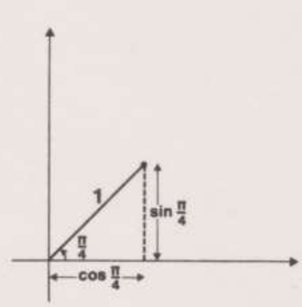
$$\tan \theta = \frac{y}{x} \qquad (x \neq 0).$$

■

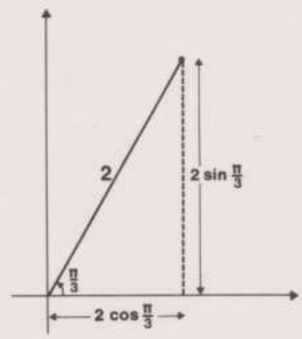
Solution 2

Solution 2

(i) The image under p is $\left(\cos \dfrac{\pi}{4}, \sin \dfrac{\pi}{4}\right) = \left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$.



(ii) The image under p is $\left(2\cos\frac{\pi}{3}, 2\sin\frac{\pi}{3}\right) = (1, \sqrt{3})$.

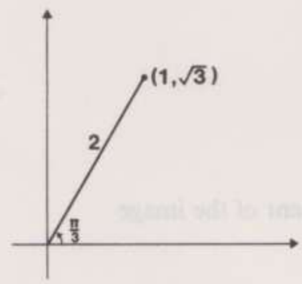


The images under P are the same for both (i) and (ii). ■

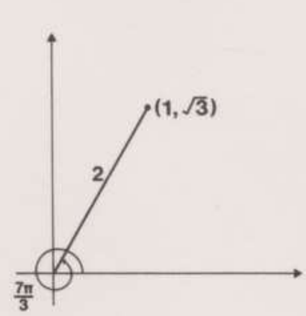
Solution 3

Solution 3

(i)

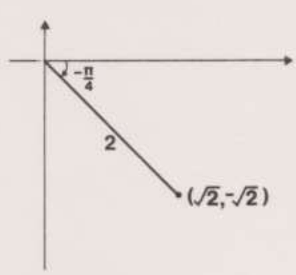


The pair $\left(2, \frac{\pi}{3}\right)$ is one image of $(1, \sqrt{3})$ under the reverse of p, but there are many more, for example, $\left(2, \frac{7\pi}{3}\right)$.



In fact there is an infinite set of images. It is $\left\{\left(2, \frac{\pi}{3} + 2k\pi\right), k \in Z\right\}$.

(ii)



You may have answered

$$\left\{\left(2, -\frac{\pi}{4} + 2k\pi\right), k \in Z\right\},$$

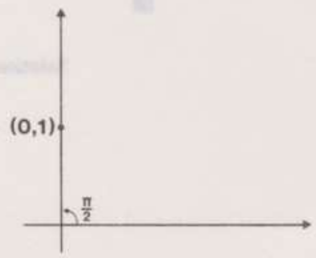
or

$$\left\{\left(2, \frac{7\pi}{4} + 2k\pi\right), k \in Z\right\};$$

both are correct.

(iii) $\{(0, \theta), \theta \in R\}$.

(iv) $\left\{\left(1, \frac{\pi}{2} + 2k\pi\right), k \in Z\right\}$.



In each case except (iii) the image under P^{-1} is the element of the image set under the reverse of p which lies in the interval $[0, 2\pi[$.
The answers are

(i) $\left(2, \frac{\pi}{3}\right)$ (ii) $\left(2, \frac{7\pi}{4}\right)$ (iv) $\left(1, \frac{\pi}{2}\right)$.

In the case of (iii) the image is $(0, 0)$. ■

27.2 "MULTIPLICATION" IN TERMS OF POLAR CO-ORDINATES

27.2

27.2.1 A New Operation on the Set of Geometric Vectors

27.2.1

Main Text

In this section we shall define a new "multiplication" operation $\circ$ on the set A, and then we shall have a look at its geometric interpretation.

We arrive at the definition in two stages. First we define such an operation on the set of all polar co-ordinates $R_0^+ \times R$. If (r_1, θ_1) and (r_2, θ_2) are any two elements of this set, then we define $\circ$ by

$$(r_1, \theta_1) \circ (r_2, \theta_2) = (r_1 r_2, \theta_1 + \theta_2).$$

If (r_1, θ_1) and (r_2, θ_2) are any two elements of the subset A of $R_0^+ \times R$, then this definition still defines a binary operation on A, but the operation is not closed. For instance,

$$(1, \pi) \circ \left(3, \frac{3\pi}{2}\right) = \left(3, \frac{5\pi}{2}\right),$$

and the latter is not an element of A. The non-closure can be a nuisance, because we shall want to restrict our attention *as much as possible* to A and the function P (as opposed to $R_0^+ \times R$ and p). So we now define $\circ$ on A by

Definition 1

$$(r_1, \theta_1) \circ (r_2, \theta_2) = (r_1 r_2, \theta_1 + \theta_2 (\text{mod } 2\pi)), \qquad (r_1, r_2 \neq 0),$$

where $\theta_1 + \theta_2(\text{mod } 2\pi)$ means addition modulo 2π.
For example, if

$$4\pi > \theta_1 + \theta_2 \geqslant 2\pi,$$

then

$$\theta_1 + \theta_2(\text{mod } 2\pi) = \theta_1 + \theta_2 - 2\pi.$$

(We could write $\theta_1 \oplus_{2\pi} \theta_2$ instead of $\theta_1 + \theta_2(\text{mod } 2\pi)$: see, for instance, *Unit 11, Logic I* for a definition of $\oplus_2$, the operation of addition modulo 2.)

We have not yet overcome the problem of closure, since, by definition,

$$A = \{(r, \theta) : r \in R^+, 0 \leqslant \theta < 2\pi\} \cup \{(0, 0)\},$$

and Definition 1 only applies to the set A_1, where

$$A_1 = \{(r, \theta) : r \in R^+, 0 \leqslant \theta < 2\pi\}.$$

So we define

$$(r, \theta) \circ (0, 0) = (0, 0) \circ (r, \theta) = (0, 0) \qquad ((r, \theta) \in R^+ \times R)$$

and

$$(0, 0) \circ (0, 0) = (0, 0).$$

The operation $\circ$ is now a closed binary operation on A.

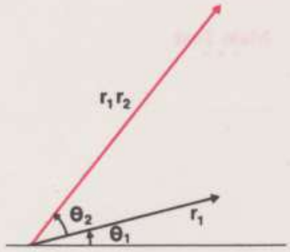
We use the same symbol for the binary operations on A and on $R_0^+ \times R$, because geometrically, say in terms of the combination of geometric vectors specified by the polar co-ordinates, the operations are the same. In fact we shall speak of one binary operation, leaving the context to make it clear, if necessary, in which set we are working. Notice that both binary operations are commutative.

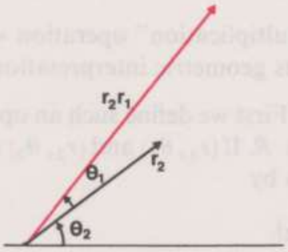
Discussion

Now let us have a look at the geometric interpretation of $\circ$.

Geometrically, this operation can be interpreted as follows: take the geometric vector determined by (r_1, θ_1), scale it up (or down) by a factor r_2, and rotate it about its blunt end-point through an angle θ_2 anti-clockwise.

Alternatively, since the operation is commutative, we can say: take the geometric vector determined by (r_2, θ_2), scale it up (or down) by a factor r_1, and rotate it about its blunt end-point through an angle θ_1 anti-clockwise.





Another interpretation is the following. If we regard (r_1, θ_1) as determining a scaling by a factor r_1 and a rotation through an angle θ_1 (as described above), then we can regard it as determining a function which maps the set of all geometric vectors to itself. (Compare *Unit 22, Linear Algebra I*, where we regarded a geometric vector as determining a translation.) In particular, the function determined by (r_1, θ_1) maps the geometric vector determined by $(1, 0)$ to the geometric vector determined by (r_1, θ_1). Then we can regard $(r_1, \theta_1) \circ (r_2, \theta_2)$ as the composition of the corresponding two functions.

The interesting thing, from our point of view, is the interpretation of the operation, which we have introduced in $R_0^+ \times R$ or A, in terms of the corresponding Cartesian co-ordinates. We have a mapping to get us from polar to Cartesian co-ordinates. We now want to turn this into a morphism by selecting the right operation in the image set.

Let (r_1, θ_1) and (r_2, θ_2) be elements of A where $r_1 \neq 0$, $r_2 \neq 0$. Then **Main Text**

$$P: (r_1, \theta_1) \longmapsto (r_1 \cos \theta_1, r_1 \sin \theta_1) = (x_1, y_1)$$

$$P: (r_2, \theta_2) \longmapsto (r_2 \cos \theta_2, r_2 \sin \theta_2) = (x_2, y_2).$$

Now

$$(r_1, \theta_1) \circ (r_2, \theta_2) = (r_1 r_2, \theta_1 + \theta_2 (\mathrm{mod}\ 2\pi)).$$

So we define the combination of (x_1, y_1) and (x_2, y_2) to correspond to the combination of (r_1, θ_1) and (r_2, θ_2); i.e. if we denote the operation on the image set by $\otimes$, then **Notation 1**

$$\begin{aligned}(x_1, y_1) \otimes (x_2, y_2) &= P((r_1 r_2, \theta_1 + \theta_2 (\mathrm{mod}\ 2\pi))) \\ &= (r_1 r_2 \cos(\theta_1 + \theta_2), r_1 r_2 \sin(\theta_1 + \theta_2)).\end{aligned}$$

(Notice that we can drop the mod 2π once we take sines and cosines.)

That is not a very useful result: we would like the right-hand side to be expressed in terms of x_1, x_2, y_1 and y_2. Notice first that

$$\begin{aligned}r_1 r_2 \cos(\theta_1 + \theta_2) &= r_1 r_2 (\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2) \\ &= x_1 x_2 - y_1 y_2.\end{aligned}$$

(See RB10)

Similarly,

$$\begin{aligned}r_1 r_2 \sin(\theta_1 + \theta_2) &= r_1 r_2 (\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2) \\ &= y_1 x_2 + x_1 y_2.\end{aligned}$$

It follows that the corresponding operation on the set of Cartesian co-ordinates, denoted by $\otimes$, is defined by

$$(x_1, y_1) \otimes (x_2, y_2) = (x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2)$$

Definition 2

This definition also covers the case when (x_1, y_1) or (x_2, y_2) is $(0, 0)$.

This formula is very important, but luckily we do not have to remember it. In section 27.3 we shall introduce a very useful notation which enables us to work out "products" quickly and easily.

Notice that $\circ$ is a simpler operation to perform than $\otimes$: this means that polar co-ordinates are easier to use when "multiplying".

Summary

Summary

In this section we have defined a multiplication operation $\circ$ on the set of polar co-ordinates; this operation is based on the ideas of scaling and rotation of geometric vectors. We have also found the induced operation $\otimes$ on the set of Cartesian co-ordinates which corresponds to $\circ$ on the set of polar co-ordinates under the morphism P.

We can summarize the way we obtained $\otimes$ from $\circ$ by drawing the commutative diagram for the morphism P for the case $r_1 \neq 0$, $r_2 \neq 0$.

$$\begin{tikzcd}
((r_1, \theta_1), (r_2, \theta_2)) \arrow[r, "\circ"] \arrow[d, "P"'] & (r_1 r_2, \theta_1 + \theta_2 (\text{mod } 2\pi)) \arrow[d, "P"] \\
((x_1, y_1), (x_2, y_2)) \arrow[r, "\otimes"] & (x_1 x_2 - y_1 y_2, y_1 x_2 + x_1 y_2)
\end{tikzcd}$$

Exercise 1

Exercise 1
(3 minutes)

Fill in the gaps in the following diagrams.

(i)
$$\begin{tikzcd}
\left(\left(1, \frac{\pi}{4}\right), \left(\frac{1}{2}, \frac{\pi}{3}\right)\right) \arrow[d, "P"'] & \\
(\qquad , \qquad) \arrow[r, "\otimes"] & (\qquad , \qquad)
\end{tikzcd}$$

(ii)
$$\begin{tikzcd}
\left(\left(1, \frac{\pi}{4}\right), \left(\frac{1}{2}, \frac{\pi}{3}\right)\right) \arrow[r, "\circ"] & (\qquad , \qquad) \arrow[d, "P"] \\
& (\qquad , \qquad)
\end{tikzcd}$$

The final answers to parts (i) and (ii) should be the same.

(iii)
$$\begin{tikzcd}
((-\sqrt{3}, 1), (-2, -2)) \arrow[d, "P^{-1}"'] & \\
(\qquad , \qquad) \arrow[r, "\circ"] & (\qquad , \qquad)
\end{tikzcd}$$

(iv)
$$\begin{tikzcd}
((-\sqrt{3}, 1), (-2, -2)) \arrow[r, "\otimes"] & (\qquad , \qquad) \arrow[d, "P^{-1}"] \\
& (\qquad , \qquad)
\end{tikzcd}$$

The final answers to parts (iii) and (iv) should be the same. ■

Solution 1

Solution 1

(i) and (ii)

$$\begin{tikzcd}
\left(\left(1, \frac{\pi}{4}\right), \left(\frac{1}{2}, \frac{\pi}{3}\right)\right) \arrow[r, "\circ"] \arrow[d, "P"'] & \left(\frac{1}{2}, \frac{7\pi}{12}\right) \arrow[d, "P"] \\
\left(\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{4}, \frac{\sqrt{3}}{4}\right)\right) \arrow[r, "\otimes"] & \left(\frac{1-\sqrt{3}}{4\sqrt{2}}, \frac{1+\sqrt{3}}{4\sqrt{2}}\right)
\end{tikzcd}$$

(iii) and (iv)

$$\begin{tikzcd}
((-\sqrt{3}, 1), (-2, -2)) \arrow[r, "\otimes"] \arrow[d, "P^{-1}"'] & (2\sqrt{3}+2, 2\sqrt{3}-2) \arrow[d, "P^{-1}"] \\
\left(\left(2, \frac{5}{6}\pi\right), \left(2\sqrt{2}, \frac{5}{4}\pi\right)\right) \arrow[r, "\circ"] & \left(4\sqrt{2}, \frac{\pi}{12}\right)
\end{tikzcd}$$

The more difficult calculation is the one on the extreme right. In (ii) we can write

$$\cos\frac{7}{12}\pi = \cos\left(\frac{\pi}{4} + \frac{\pi}{3}\right) = \cos\frac{\pi}{4}\cos\frac{\pi}{3} - \sin\frac{\pi}{4}\sin\frac{\pi}{3} = \frac{1-\sqrt{3}}{2\sqrt{2}},$$

(See RB10)

$$\sin\frac{7}{12}\pi = \sin\left(\frac{\pi}{4} + \frac{\pi}{3}\right) = \sin\frac{\pi}{4}\cos\frac{\pi}{3} + \cos\frac{\pi}{4}\sin\frac{\pi}{3} = \frac{1+\sqrt{3}}{2\sqrt{2}}.$$

In (iv) we have to calculate

$$\begin{aligned} r &= \sqrt{(2\sqrt{3}+2)^2 + (2\sqrt{3}-2)^2} \\ &= \sqrt{12 + 4 + 8\sqrt{3} + 12 + 4 - 8\sqrt{3}} \\ &= \sqrt{32} = 4\sqrt{2}, \end{aligned}$$

and θ from

$$\cos\theta = \frac{2\sqrt{3}+2}{4\sqrt{2}} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$$\sin\theta = \frac{2\sqrt{3}-2}{4\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

These expressions need simplification, conversion to decimal form and then tables to find the angle $\theta = 15° = \frac{\pi}{12}$; alternatively, simplification using trigonometric identities can be used. ■

27.2.2 The Argument

27.2.2

Main Text

We know that the mapping

$$p:(r, \theta) \longmapsto (x, y) \qquad ((r, \theta) \in R_0^+ \times R)$$

is many-one. For a given value of $r \neq 0$ there are many different angles θ which will map to the same pair (x, y). Each of these angles is called a *value of the argument* of (x, y). The argument is itself the set of *all* such values, so that

Definition 1

$$\arg(x, y)$$

is the set

$$\{\theta : p((r, \theta)) = (x, y)\}.$$

If θ_1 is any particular angle lying in this set, then

$$\arg(x, y) = \{\theta_1 + 2k\pi, k \in Z\}.$$

(The argument is sometimes called the *amplitude*).

$$\text{If } P:(r, \theta) \longmapsto (x, y), \qquad (r \neq 0)$$

then we say that the principal value of the argument of (x, y) is θ, and we denote it by $\operatorname{Arg}(x, y)$, so

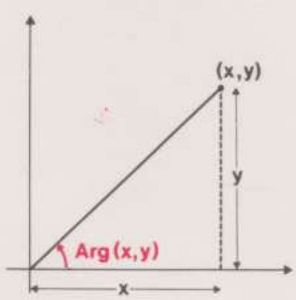
Definition 2

$$\operatorname{Arg}(x, y) = \theta.$$

We can regard the principal value of the argument as defining a function

$$\operatorname{Arg}:(x, y) \longmapsto \theta \qquad ((x, y) \in R \times R, (x, y) \neq (0, 0)).$$

Notice that $\operatorname{Arg}(x, y)$ is simply the element of $\arg(x, y)$ which lies in the interval $[0, 2\pi[$, and P^{-1} picks out this element from the set of all possible angles in the set $\arg(x, y)$.



We have seen in *Unit 19, Relations* that any many-one mapping defines an equivalence relation on its domain: this is exactly what has happened with p. Given an (x, y), the set of all θ's for which $p:(r, \theta) \longmapsto (x, y)$ forms an equivalence class*, and we call this class $\arg(x, y)$. The mapping P^{-1} gives us a way of choosing representatives from each of the equivalence classes.

Exercise 1

Exercise 1
(3 minutes)

Find the argument and its principal value for each of the following:

(i) $(1, \sqrt{3})$

(ii) $(\sqrt{2}, -\sqrt{2})$

(iii) $(0, 1)$

(You will be able to use the results of Exercise 27.1.2.3.) ■

* Actually the equivalence class is the set of pairs (r, θ), but since the r in each pair in the equivalence class is the same, we have allowed ourselves mathematical licence.

Solution 1

Solution 1

(i) $\arg(1, \sqrt{3}) = \left\{\frac{\pi}{3} + 2k\pi, k \in Z\right\}$

$\operatorname{Arg}(1, \sqrt{3}) = \frac{\pi}{3}$

(ii) $\arg(\sqrt{2}, -\sqrt{2}) = \left\{-\frac{\pi}{4} + 2k\pi, k \in Z\right\}$

$\operatorname{Arg}(\sqrt{2}, -\sqrt{2}) = \frac{7}{4}\pi$

(iii) $\arg(0, 1) = \left\{\frac{\pi}{2} + 2k\pi, k \in Z\right\}$

$\operatorname{Arg}(0, 1) = \frac{\pi}{2}$ ■

Discussion
* *

The next question is fairly natural. We have defined a function Arg on a set on which we have a binary operation $\otimes$. Is there an operation $\square$ such that

$$\operatorname{Arg}((x_1, y_1) \otimes (x_2, y_2)) = \operatorname{Arg}(x_1, y_1) \square \operatorname{Arg}(x_2, y_2)?$$

In fact, although arg is not a function, we shall ask a more general question: What is the argument of $(x_1, y_1) \otimes (x_2, y_2)$ in terms of $\arg(x_1, y_1)$ and $\arg(x_2, y_2)$? (We assume that these arguments exist.)

Suppose that we have the following situation:

Polar Co-ordinates		Cartesian Co-ordinates
(r_1, θ_1)	$\xmapsto{p}$	(x_1, y_1)
(r_2, θ_2)	$\xmapsto{p}$	(x_2, y_2)

i.e.

$$(r_1 r_2, \theta_1 + \theta_2) \xmapsto{p} (x_1, y_1) \otimes (x_2, y_2)$$

We therefore know that if $r_1 \neq 0$ and $r_2 \neq 0$, then

$$\arg(x_1, y_1) = \{\theta_1 + 2k\pi, k \in Z\},$$

$$\arg(x_2, y_2) = \{\theta_2 + 2k\pi, k \in Z\}$$

and also

$$\arg((x_1, y_1) \otimes (x_2, y_2)) = \{\theta_1 + \theta_2 + 2k\pi, k \in Z\}.$$

It follows that we can obtain $\arg((x_1, y_1) \otimes (x_2, y_2))$ from $\arg(x_1, y_1)$ and $\arg(x_2, y_2)$ by a sort of addition: we can, for instance, add each of the elements of $\arg(x_1, y_1)$ to each of the elements of $\arg(x_2, y_2)$. It would be simpler to add *one* element of $\arg(x_1, y_1)$ to each of the elements of $\arg(x_2, y_2)$ or vice versa.

Exercise 2

Exercise 2
(4 minutes)

If $(x_1, y_1) \neq (0, 0)$ and $(x_2, y_2) \neq (0, 0)$, what is $\square$ in

$$\operatorname{Arg}((x_1, y_1) \otimes (x_2, y_2)) = \operatorname{Arg}(x_1, y_1) \square \operatorname{Arg}(x_2, y_2)?$$ ■

Summary

Summary

In this section we have defined the mappings Arg and arg which respectively associate an angle and a set of angles with the pair of Cartesian co-ordinates (x, y). $\operatorname{Arg}(x, y)$ is the angle (measured anti-clockwise) between the positive x-axis and the straight line from the origin to the point (x, y). We have also found the induced operation $\square$ on the set of Arg's which corresponds to the operation $\otimes$ on the set of Cartesian co-ordinates without $(0, 0)$.

Solution 2

Solution 2

Looking back at the table of corresponding polar and Cartesian co-ordinates, and replacing p by P, it is tempting to say that $\square$ is $+$. But this is not quite true. For instance, consider

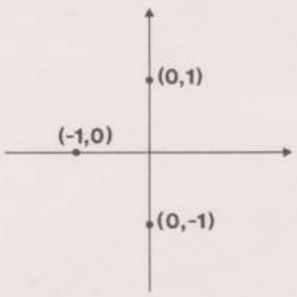
$$(x_1, y_1) = (-1, 0)$$

and

$$(x_2, y_2) = (0, -1)$$

then

$$(x_1, y_1) \otimes (x_2, y_2) = (0, 1)$$



$\mathrm{Arg}(-1, 0) = \pi$ and $\mathrm{Arg}(0, -1) = \dfrac{3\pi}{2}$.

So their sum is $\dfrac{5\pi}{2}$, *but* $\mathrm{Arg}(0, 1) = \dfrac{\pi}{2}$.

In fact, our previous experience (when we introduced the binary operation for the set A_1) should tell us that $\square$ is $\oplus_{2\pi}$, i.e. addition modulo 2π. ■

27.3 THE SET OF COMPLEX NUMBERS

27.3

27.3.1 Real and Complex Numbers

27.3.1

Let us now return to the problem we posed in the Introduction to the text: finding a new domain for the "square" function, so that we can find an element in the domain which maps to -1. **Definitions** * * *

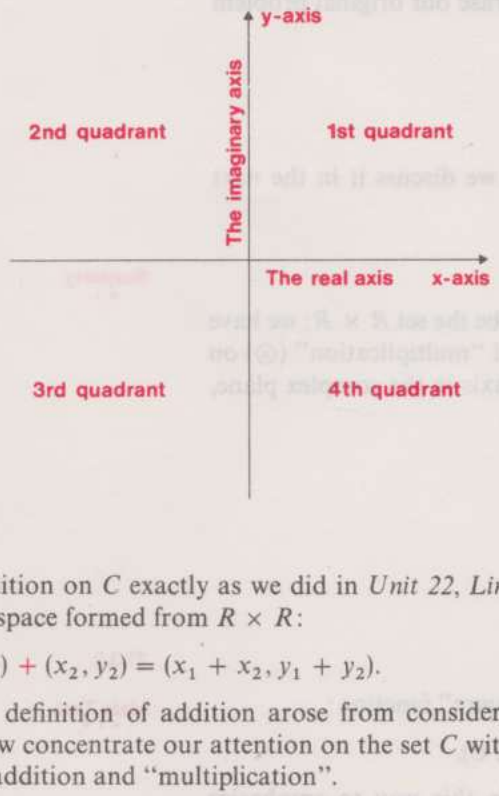
The set which we shall take as the new domain of our "square" function is the set of pairs of real numbers (x, y). We denote* this set by C; for the operation of "multiplication" on C we shall use $\otimes$. **Notation 1** * * *

We shall call the elements of C complex numbers; that is, each complex number is in fact an ordered pair of real numbers. In order to distinguish the two numbers in this pair (and for historical reasons), we call the first number the real part and the second number the imaginary part of the complex number. **Definition 1** * * * **Definition 2** * * *

It is often useful to plot complex numbers (x, y) on a graph in the usual way. Such a graph in this context is called an Argand† diagram. **Definition 3** * * *

On an Argand diagram, the set $\{(x, 0), x \in R\}$ is represented by the x-axis, and, since we shall identify this set with the set of real numbers, this line is often called the real axis. The y-axis, which represents the points $\{(0, y), y \in R\}$, is often called the imaginary axis, and the set of points representing C is often called the complex plane. **Definition 4** * * * **Definition 5** * * * **Definition 6** * * *

The complex plane



We define addition on C exactly as we did in *Unit 22, Linear Algebra I* for the vector space formed from $R \times R$:

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

Definition 7 * * *

Although our definition of addition arose from considering geometric vectors, we now concentrate our attention on the set C with its algebraic operations of addition and "multiplication". **Discussion** * *

One of the requirements of our original discussion was that the new domain of the "square" function should contain R as a subset. Strictly speaking, R is not a subset of C, but consider the subset

$$D = \{(x, 0), x \in R\}.$$

* C is, of course, $R \times R$, so why another name? The reason is that ordered pairs of numbers are used in many contexts. We have a particular context here, so we use a symbol for the set of ordered pairs which automatically indicates the context.

† This graphical representation was apparently suggested by the Norwegian surveyor Casper Wessel (1797) and later by several authors including J. R. Argand (1806) and Gauss.

Addition in D takes the form

$$(x_1, 0) + (x_2, 0) = (x_1 + x_2, 0),$$

and this corresponds exactly to our normal addition of real numbers. If we define a one-one function:

$$f:(x, 0) \longmapsto x \qquad ((x, 0) \in D),$$

then we can draw the following commutative diagram.

$$\begin{tikzcd}
((x_1, 0), (x_2, 0)) \arrow[r, "+(\text{in } C)"] \arrow[d, "f"'] & (x_1 + x_2, 0) \arrow[d, "f"] \\
(x_1, x_2) \arrow[r, "+(\text{in } R)"] & x_1 + x_2
\end{tikzcd}$$

Since $(x_1, 0) \otimes (x_2, 0) = (x_1 x_2, 0)$, we also have

$$\begin{tikzcd}
((x_1, 0), (x_2, 0)) \arrow[r, "\otimes"] \arrow[d, "f"'] & (x_1 x_2, 0) \arrow[d, "f"] \\
(x_1, x_2) \arrow[r, "\times"] & x_1 \times x_2
\end{tikzcd}$$

So f is an isomorphism from D to R for both the addition and multiplication operations. Although we do not have R as a subset of our new domain C, we do have D in which the arithmetic is the same as the arithmetic in R. It is not difficult to rephrase problems in R in terms of D. Since $(-1, 0)$ in C corresponds to -1 in R, we can now rephrase our original problem in the following form:

Is there an element $(x, y) \in C$ such that

$$(x, y) \otimes (x, y) = (-1, 0)?$$

You may now be able to solve this problem; we discuss it in the next section.

Summary — Summary

We have defined the set of complex numbers to be the set $R \times R$; we have also defined the operations of addition ($+$) and "multiplication" ($\otimes$) on this set. The set R can be identified with the x-axis in the complex plane, that is, the set D. We have an isomorphism:

$$f:(D, +, \otimes) \longmapsto (R, +, \times).$$

27.3.2 The "Square" Function

27.3.2

Main Text

We are now in a position to define our new "square" function:

$$\text{sq}:(x, y) \longmapsto (x, y) \otimes (x, y), \qquad ((x, y) \in C).$$

We have written the right-hand expression in this way to emphasize the "square", but from our definition we know that

$$(x, y) \otimes (x, y) = (x^2 - y^2, 2xy),$$

so that

$$\text{sq}:(x, y) \longmapsto (x^2 - y^2, 2xy), \qquad ((x, y) \in C).$$

Notice that sq maps C to C, and, although sq does not look much like our well known real "square" function ($x \longmapsto x^2, x \in R$), the two functions do have some interesting and, in fact, vital things in common.

We already know that

$$\text{sq}:(x,0) \longmapsto (x^2,0) \qquad (x \in R)$$

so the restriction of sq to the subset D is almost the same as the real "square" function.

Now for the crucial question. Is there an element of C which maps to $(-1,0)$ under sq? In other words, can we choose (x, y) in such a way that

$$\text{sq}\,(x,y) = (-1,0)?$$

(Remember that we are identifying $(-1,0)$ with -1.) We know that

$$\text{sq}\,(x,y) = (x^2 - y^2, 2xy)$$

and number pairs can only be equal if the corresponding elements are equal. We require that

$$(x^2 - y^2, 2xy) = (-1,0),$$

that is,

$$x^2 - y^2 = -1$$

and

$$2xy = 0.$$

The second equation implies that either $x = 0$ or $y = 0$. If $y = 0$, then the first equation cannot possibly be true for any real x. On the other hand, $x = 0$ implies that $y = \pm 1$. We have therefore shown that

$$\text{sq}:(0,1) \longmapsto (-1,0)$$

and

$$\text{sq}:(0,-1) \longmapsto (-1,0).$$

The complex numbers $(0,-1)$ and $(0,1)$ are the "square roots" of $(-1,0)$, and we are gratified to find that there are two "square roots", just as there are two real square roots of any positive real number.

27.3.3 A Useful Notation

27.3.3

Notation

The development of complex numbers so far in this text has been aimed at giving a firm base on which to build, but the notation which we have used is not very practical. We shall now introduce a notation which is a considerable aid to computation.

We can rewrite the complex number

$$(x,y) = (x,0) + (0,y)$$

in the form $x + iy$, or sometimes $x + yi$, where the letter i is used merely as a notational device to indicate that y is the second number in the original pair. As we have noted before, x is called the *real part* of the complex number and y is called the *imaginary part*.* The rule for "multiplication" of complex numbers: Notation 1

$$(x_1, y_1) \otimes (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1)$$

then becomes

$$(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1),$$

if we follow the convention of the algebra of real numbers and drop the special symbol $\otimes$ for "multiplication". (Occasionally, in the following sections, we shall insert the symbol $\otimes$ when we wish to emphasize its use.)

* It is a common mistake to say that the imaginary part is iy rather than y.

You can easily check that if we multiply out the left-hand side of the last equation as for ordinary real algebra, and whenever we encounter $i \times i$ we replace it by -1, then we get the right-hand side.

We emphasize that there is no suggestion that i is some sort of distorted real number; it is simply a device for separating the two parts of a complex number. However, we can use ordinary algebraic rules when manipulating elements like $x + iy$, and this justifies calling them "complex" numbers. It is common practice to represent $x + iy$ by z for convenience. We shall denote the real and imaginary parts of the complex number z by $\text{Re}\, z$ and $\text{Im}\, z$ respectively; that is,

Notation 2

Notation 3

$$\text{Re}\, z = x$$

and

$$\text{Im}\, z = y.$$

We have already seen the mapping

$$P^{-1}: z \longmapsto (r, \theta) \qquad z \in C$$

(but in the slightly different form $(x, y) \longmapsto (r, \theta)$), where $r = \sqrt{x^2 + y^2}$ and $\theta = \text{Arg}\, z$.

We know that $x = r \cos\theta$ and $y = r \sin\theta$. If we write

$$z = x + iy = r\cos\theta + ir\sin\theta$$

i.e.

$$z = r(\cos\theta + i\sin\theta),$$

then we say that the right-hand expression is the polar form of the complex number z.

Notation 4

We shall write $z \times z$ as z^2, and so on.

Exercise 1

Exercise 1
(3 minutes)

(i) Simplify each of the following into the form $x + iy$:

(a) $(1 + i)(2 + i)(2 - i)$
(b) $(1 + 2i)(1 - 2i)$
(c) $1 + 3i(4 + 5i) + (2 - i)(1 + i)$

(ii) If $z_1 = 1 + 3i$ and $z_2 = 2 - i$, evaluate $z_1^2 z_2$.

(iii) Plot the points corresponding to $1 + 3i$ and $2 - i$ on an Argand diagram. ■

Exercise 2

Exercise 2
(3 minutes)

Find:

(i) $\text{Re}\,((1 + 2i)^2)$ and $\text{Im}\,((1 + 2i)^2)$;
(ii) $\text{Re}\,(1 + 2i + 3i^2 + i^3)$ and $\text{Im}\,(1 + 2i + 3i^2 + i^3)$;
(iii) $\text{Arg}\,(1 + i)$ and $\text{Arg}\,((1 + i)^2)$. ■

Exercise 3

Exercise 3
(1 minute)

Let $\quad z_1 = x_1 + iy_1,$

$$z_2 = x_2 + iy_2,$$

$$z_3 = x_3 + iy_3.$$

Show that

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3$$

and that

$$(z_2 + z_3)z_1 = z_2 z_1 + z_3 z_1.$$ ■

Exercise 4

Exercise 4
(1 minute)

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$,

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$,

$z_3 = r_3(\cos\theta_3 + i\sin\theta_3)$.

Show that

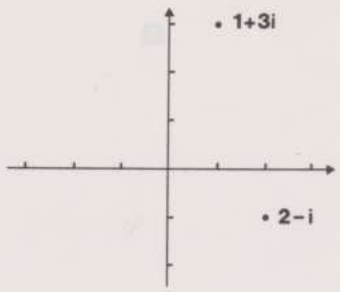
$$z_1(z_2 z_3) = (z_1 z_2)z_3.$$

■

Solution 1

Solution 1

(i) (a) $5 + 5i$
   (b) 5
   (c) $-11 + 13i$
(ii) $-10 + 20i$
(iii)

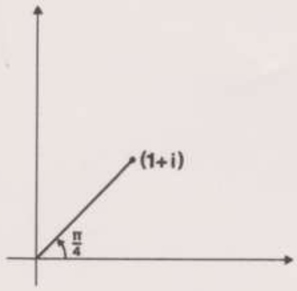


■

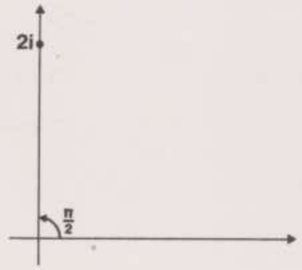
Solution 2

Solution 2

(i) $(1 + 2i)^2 = (1 + 2i)(1 + 2i) = 1 + 4i + 4i^2 = 1 + 4i - 4 = -3 + 4i$. Hence $\mathrm{Re}\,((1 + 2i)^2) = -3$. Also $\mathrm{Im}\,((1 + 2i)^2) = 4$. Notice particularly that the answer is 4, not $4i$.
(ii) $1 + 2i + 3i^2 + i^3 = 1 + 2i - 3 - i = -2 + i$. The real part is -2; the imaginary part is 1.
(iii)



$$\mathrm{Arg}\,(1 + i) = \frac{\pi}{4}.$$



$$\mathrm{Arg}\,(1 + i)^2 = \frac{\pi}{2}.$$

■

Solution 3 **Solution 3**

$$\begin{aligned}
z_1(z_2 + z_3) &= (x_1 + iy_1)((x_2 + iy_2) + (x_3 + iy_3)) \\
&= (x_1 + iy_1)((x_2 + x_3) + i(y_2 + y_3)) \\
&= (x_1(x_2 + x_3) - y_1(y_2 + y_3)) \\
&\quad + i(x_1(y_2 + y_3) + y_1(x_2 + x_3)) \\
&= (x_1x_2 + x_1x_3 - y_1y_2 - y_1y_3) \\
&\quad + i(x_1y_2 + x_1y_3 + y_1x_2 + y_1x_3) \\
&= ((x_1x_2 - y_1y_2) + i(x_1y_2 + y_1x_2)) \\
&\quad + ((x_1x_3 - y_1y_3) + i(x_1y_3 + y_1x_3)) \\
&= z_1z_2 + z_1z_3.
\end{aligned}$$

Since multiplication of complex numbers is commutative,

$$\begin{aligned}
(z_2 + z_3)z_1 &= z_1(z_2 + z_3) \\
&= z_1z_2 + z_1z_3 \\
&= z_2z_1 + z_3z_1.
\end{aligned}$$

Hence multiplication is distributive over addition. Note that we have used the associative and distributive properties of the real numbers. ■

Solution 4 **Solution 4**

$$\begin{aligned}
z_1(z_2z_3) &= r_1(\cos\theta_1 + i\sin\theta_1) \\
&\quad \times (r_2r_3(\cos(\theta_2 + \theta_3) + i\sin(\theta_2 + \theta_3))) \\
&= r_1r_2r_3(\cos(\theta_1 + \theta_2 + \theta_3) + i\sin(\theta_1 + \theta_2 + \theta_3)) \\
&= r_1r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)) \\
&\quad \times r_3(\cos\theta_3 + i\sin\theta_3) \\
&= (z_1z_2)z_3.
\end{aligned}$$

Hence multiplication is associative.

Note that we have used the associative and distributive properties of the real numbers. Also, the proof of the associativity of multiplication is easier if we use polar co-ordinates as opposed to Cartesian co-ordinates. ■

27.3.4 Summary of Properties of Complex Numbers

27.3.4

Discussion

**

We began with the ancient problem of defining $\sqrt{-1}$, and, by extending the domain of the "square" function to the set of complex numbers, we were able to find elements which map to $(-1, 0)$. In our new notation we could replace $(-1, 0)$ by $-1 + i0$ so that

$$(0 + i)^2 = -1 + i0,$$

and

$$(0 - i)^2 = -1 + i0.$$

Normally we would simplify these expressions still further and write

$$i^2 = -1$$

and

$$(-i)^2 = -1.$$

Such abbreviation suggests the commonly used (but suspect) statement that i and $-i$ are the square roots of -1.

We have in fact done more than simply examine a piece of mathematical history. The system which we have developed is a powerful extension of the algebra of real numbers. There are many strategic advantages to be gained from extending our number system to include the complex numbers, if only because the real number system is, in a sense, incomplete. For example, it is true that a polynomial equation of degree n has n complex solutions (some of which may coincide), but, for instance, the polynomial equation $x^2 + 1 = 0$ has no real solutions.

Eliminating complex numbers from mathematics and its applications today would have almost as drastic an effect as eliminating the negative numbers.

Some of the properties of complex numbers are listed below.

Summary

(i) $x + iy$ is simply a convenient way of writing the complex number (x, y).

(ii) $x_1 + iy_1 = x_2 + iy_2$ if and only if $x_1 = x_2$ and $y_1 = y_2$.

(iii) $(x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$
(the complex numbers are closed for addition)

(iv) $z_1 + z_2 = z_2 + z_1$
(addition is commutative)

(v) $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$
(addition is associative)

(vi) $(x_1 + iy_1)(x_2 + iy_2) = (x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)$
(the complex numbers are closed for multiplication)

(vii) $z_1z_2 = z_2z_1$
(multiplication is commutative)

(viii) $z_1(z_2z_3) = (z_1z_2)z_3$
(multiplication is associative)

(ix) $z_1(z_2 + z_3) = z_1z_2 + z_1z_3$
$(z_2 + z_3)z_1 = z_2z_1 + z_3z_1$
(multiplication is distributive over addition)

(x) There are two complex numbers, $0 + 0i$ and $1 + 0i$ (which are not equal), with the properties that, for any complex number z,

$$z + (0 + 0i) = z \quad \text{and} \quad z(1 + 0i) = z.$$

You should compare this list of properties with the properties Re (1), Re (2) and Re (3) of the real numbers, listed on page 1 of *Unit 6, Inequalities*.

Notice particularly that the complex numbers are closed for multiplication. You will recall that we constructed a generalization of "multiplication" in *Unit 22, Linear Algebra I* which we called the *inner product*, and we noticed that it was *not* a closed binary operation on the set of geometric vectors, and therefore it was difficult to define an extension of division adequately. On this occasion that difficulty does not arise.

We have seen effectively three ways of representing a complex number:

(1) in its Cartesian form (x, y) (which we often prefer to write as $z = x + iy$);

(2) in polar form (r, θ);

(3) as a point on an Argand diagram.

Often we switch from one representation to the other, and, although we ought to distinguish between them, we may sometimes refer to "the point $x + iy$", or we might say, for example, that "a complex number lies on a straight line drawn between two other complex numbers". In other words, we use the representation which most suits our purpose, without going into a lengthy explanation each time.

We close this section with some exercises which are intended to give you practice in manipulating complex numbers, and which also indicate some general points.

Exercise 1

Exercise 1
(2 minutes)

Show that the solution set of the equation

$$x^2 + 2x + 4 = 0 \qquad (x \in R)$$

is empty. Show that, if we rewrite this equation as an equation in C, i.e.

$$z^2 + (2, 0)z + (4, 0) = (0, 0) \qquad (z \in C),$$

which we shall write as

$$z^2 + 2z + 4 = 0 \qquad (z \in C),$$

then the solution set is $\{-1 + i\sqrt{3}, -1 - i\sqrt{3}\}$. ■

Exercise 2

Exercise 2
(2 minutes)

(i) Plot the point $3 + 4i$ on an Argand diagram, then successively plot the points

$$i(3 + 4i), i^2(3 + 4i), i^3(3 + 4i), i^4(3 + 4i).$$

(ii) (a) What geometric effect does multiplying a complex number by $1 + i$ have?
(b) Plot $1 + i$ on an Argand diagram, then plot $(1 + i)^2$ and interpret the result as a magnification and rotation of the geometric vector $1 + i$.
(c) Do the same for $(1 + i)^3$.
Express $(1 + i)^{10}$ in the form $x + iy$. ■

Exercise 3

Exercise 3
(2 minutes)

Show that

(i) $(1 + i)(\frac{1}{2} - \frac{1}{2}i) = 1$
(ii) $(3 + 4i)(\frac{3}{25} - \frac{4}{25}i) = 1$
(iii) $(a + ib)(a - ib) = a^2 + b^2$.

(Notice that, as mentioned above, we are abbreviating $x + 0i$ to x and therefore we have written 1 and $a^2 + b^2$ on the right-hand sides.) ■

Exercise 4

Exercise 4
(2 minutes)

If $z = r(\cos\theta + i\sin\theta)$, where $r \neq 0$, what is $\arg z$? ■

Solution 1 **Solution 1**

The fact that the equation in R has an empty solution set can be deduced in many ways (see RB5 if necessary). To show that the given elements satisfy the equation in C is a matter of calculation. For instance, we simply substitute $-1 + i\sqrt{3}$ into the left-hand side to obtain

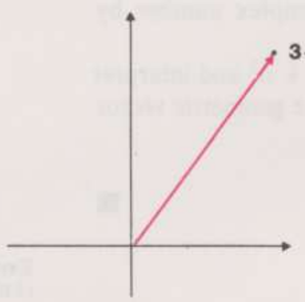
$$(-1 + i\sqrt{3})^2 + 2(-1 + i\sqrt{3}) + 4$$

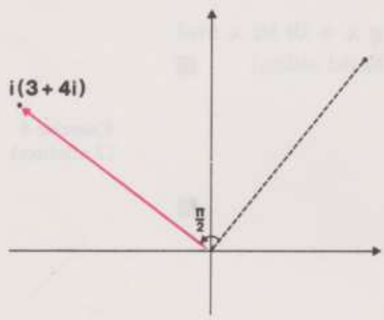
and this simplifies to zero.

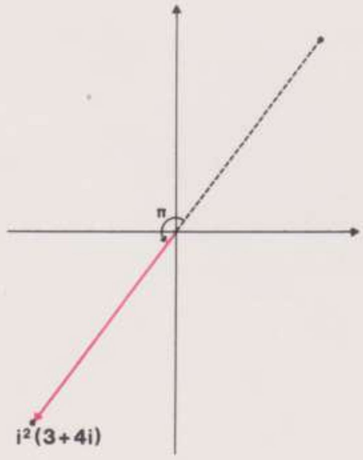
We really also need to show that there are no other elements in the solution set, but we leave this for the time being. ■

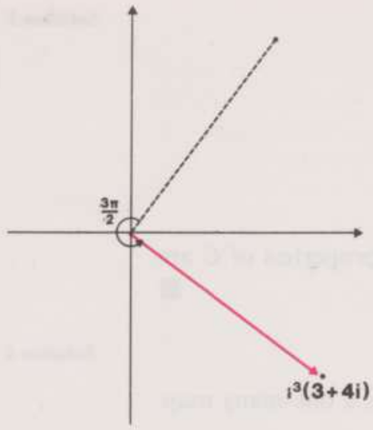
Solution 2 **Solution 2**

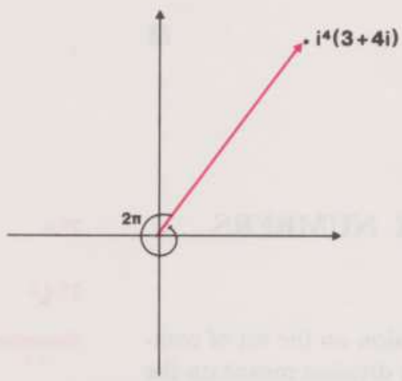
(i)









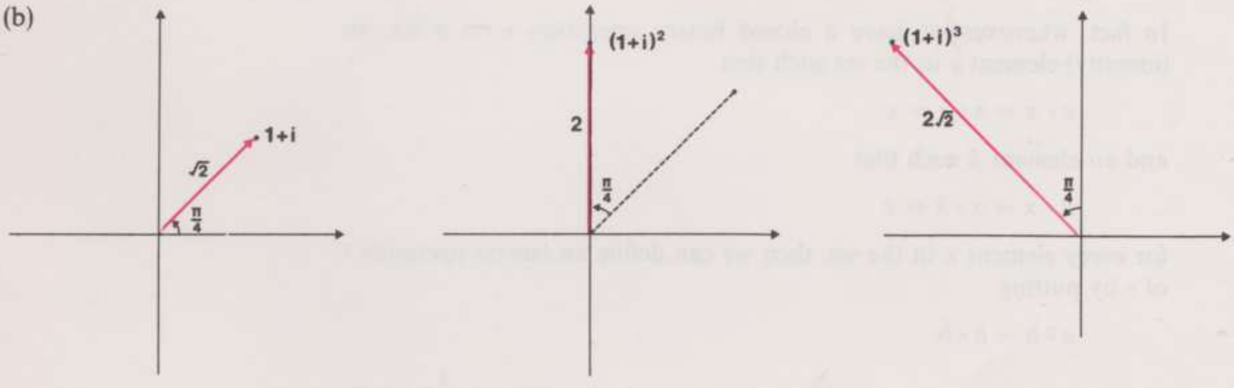


Notice that each time we multiply by i the corresponding geometric vector is rotated about its blunt end-point through $\frac{\pi}{2}$ anti-clockwise.

(ii) (a) The corresponding geometric vector is rotated about its blunt end-point through an angle $\operatorname{Arg}(1 + i) = \frac{\pi}{4}$ anti-clockwise, and magnified by a factor of $\sqrt{2}$, since

$$P:\left(\sqrt{2}, \frac{\pi}{4}\right) \longmapsto (1, 1).$$

(b)



(c) It is clear that $(1 + i)^{10}$ can be obtained by rotating the geometric vector with Cartesian co-ordinates $(1, 0)$ through $\frac{\pi}{4}$ ten times, and magnifying it by a factor of $\sqrt{2}$ ten times. The resulting geometric vector has polar co-ordinates

$$\left((\sqrt{2})^{10}, \frac{10\pi}{4}\right) \quad \text{or} \quad \left(32, \frac{\pi}{2}\right),$$

and its Cartesian co-ordinates are therefore $(0, 32)$, so that the resulting complex number is $32i$. ■

Solution 3

Solution 3

(i) $(1 + i)(\frac{1}{2} - \frac{1}{2}i) = \frac{1}{2} - \frac{1}{2}i + \frac{1}{2}i - \frac{1}{2}i^2$
$= \frac{1}{2} + \frac{1}{2} = 1$

(ii) $(3 + 4i)(\frac{3}{25} - \frac{4}{25}i) = \frac{9}{25} - \frac{12}{25}i + \frac{12}{25}i - \frac{16}{25}i^2$
$= \frac{9}{25} + \frac{16}{25} = 1$

(iii) $(a + ib)(a - ib) = a^2 - aib + iba - i^2b^2$
$= a^2 + b^2.$

(You should make sure that you understand which properties of C are being used at each step in the above manipulations.) ■

Solution 4

Solution 4

If you answered θ, then you have forgotten that arg is a one-many map. The correct answer is

$$\{\theta + 2k\pi, k \in Z\}$$ ■

27.4 THE ALGEBRA OF COMPLEX NUMBERS

27.4

27.4.1 Division

27.4.1

Discussion

In this section we shall investigate the idea of division on the set of complex numbers. First let us understand clearly what division means on the set R. Suppose that we know how to multiply real numbers; then for any non-zero real number q we know that

$$q \times \frac{1}{q} = 1.$$

We can define division by

$$p \div q = p \times \frac{1}{q},$$

where $p \in R$.

In fact, whenever we have a closed binary operation $\circ$ on a set, an (identity) element e in the set such that

$$e \circ x = x \circ e = x$$

and an element $\tilde{x}$ such that

$$\tilde{x} \circ x = x \circ \tilde{x} = e$$

for every element x in the set, then we can define an *inverse operation* $\tilde{\circ}$ of $\circ$ by putting

$$a \,\tilde{\circ}\, b = a \circ \tilde{b}.$$

For real numbers we could take $\circ$ to be $\times$, e to be 1 and $\tilde{x}$ to be $\frac{1}{x}$ $(x \neq 0)$; then the operation $\tilde{\circ}$ so defined is simply $\div$. (This idea will be discussed in *Unit 30, Groups I*.)

Let us follow exactly the same reasoning: given a complex number $\alpha = a + ib$, our first task is to find another complex number $z = x + iy$ such that $\alpha z = 1$.

Much of the following work is given in the form of exercises, since most of it is a straightforward development of concepts you have already met. You should therefore work the exercises as you come to them.

Exercise 1

Exercise 1
(1 minute)

If

$$(a + ib)(x + iy) = 1,$$

show that

$$x = \frac{a}{a^2 + b^2} \quad \text{and} \quad y = \frac{-b}{a^2 + b^2}.$$

Are there any complex numbers $a + ib$ for which such a number $x + iy$ does not exist? ■

Main Text

From the last exercise we see that a sensible definition of $\frac{1}{\alpha}$ is

$$\frac{a - ib}{a^2 + b^2}$$

provided that $a + ib \neq 0$.

Definition 1

We can now define division of complex numbers by

$$z_1 \div z_2 = z_1 \otimes \frac{1}{z_2} \qquad (z_2 \neq 0).$$

(Instead of $z_1 \div z_2$, we often write $\frac{z_1}{z_2}$.) This definition leads us to define two further terms in connection with complex numbers; we discuss them in the next section.

Solution 1

Solution 1

We are given that

$$(a + ib)(x + iy) = 1,$$

and therefore

$$(ax - by) + i(bx + ay) = 1,$$

so that

$$ax - by = 1$$

and

$$bx + ay = 0.$$

Hence (multiplying the first equation by a and the second by b, and adding)

$$(a^2 + b^2)x = a$$

so that

$$x = \frac{a}{a^2 + b^2},$$

provided that $a^2 + b^2 \neq 0$.

In other words, $a \neq 0$ *and* $b \neq 0$, so that we cannot allow $a + ib = 0$.

Similarly, $y = \dfrac{-b}{a^2 + b^2}$ provided that $a + ib \neq 0$.

So the only number for which $x + iy$ does not exist is 0, and this corresponds exactly to our experience with real numbers. ■

27.4.2 Complex Conjugate and Modulus

27.4.2
Main Text

If

$$z = x + iy \qquad (z \neq 0),$$

then we have defined $\frac{1}{z}$ to be

$$\frac{x - iy}{x^2 + y^2}.$$

Definition 1

If $z = x + iy$, then we define $x - iy$ to be the conjugate of z, which we denote by $\bar{z}$. (We read $\bar{z}$ as "z bar".)

i.e. $\quad \bar{z} = x - iy.$

Definition 2

If $z = x + iy$, then we define the positive (or zero) number $r = \sqrt{x^2 + y^2}$ to be the modulus of z and we denote it by $|z|$. (We read $|z|$ as "mod z".)

i.e. $\quad |z| = \sqrt{x^2 + y^2}.$

Notice that this is consistent with our definition of the modulus of a real number, for if $y = 0$, then $|z| = \sqrt{x^2} = |x|$.

Example 1

Example 1

If $z_1 = 2 + 3i$, $z_2 = 2 - 3i$, $z_3 = 3$, $z_4 = -3i$,

then

$$z_2 = \bar{z}_1$$
$$z_1 = \bar{z}_2$$
$$\bar{z}_3 = 3$$
$$\bar{z}_4 = 3i$$
$$|z_1| = \sqrt{2^2 + 3^2} = \sqrt{13} = |z_2|$$
$$|z_3| = \sqrt{3^2} = 3 = |z_4|.$$

■

Exercise 1

Exercise 1
(2 minutes)

Verify the following results given in terms of the new definitions.

(i) $z\bar{z} = |z|^2$

(ii) $\dfrac{1}{z} = \dfrac{\bar{z}}{|z|^2} \qquad (z \neq 0)$

(iii) $z_1 \div z_2 = z_1 \times \dfrac{\bar{z}_2}{|z_2|^2} \qquad (z_2 \neq 0)$

(iv) $z + \bar{z} = 2\,\mathrm{Re}\, z$

(v) $z - \bar{z} = 2i\,\mathrm{Im}\, z.$

■

Exercise 2

Exercise 2
(4 minutes)

(i) Represent z and $\bar{z}$ on an Argand diagram for a general complex number $z = x + iy$.

(ii) Give a geometric interpretation of $|z_1 - z_2|$, where z_1 and z_2 are any complex numbers.

■

Exercise 3

Exercise 3
(3 minutes)

Reduce each of the following expressions to the form $x + iy$:

(i) $\dfrac{1 - i}{3 + i}$ $\qquad$ (ii) $\dfrac{1}{1 + i} + \dfrac{1 + i}{i}.$

■

Solution 1 **Solution 1**

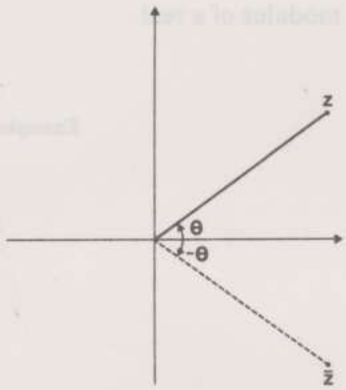
(i) $z\bar{z} = (x + iy)(x - iy) = x^2 - ixy + iyx - i^2y^2 = x^2 + y^2 = |z|^2$

(ii) $\dfrac{1}{z} = \dfrac{x - iy}{x^2 + y^2} = \dfrac{\bar{z}}{|z|^2}$

(iii) follows from (ii) and the definition of $\div$.
(iv) $z + \bar{z} = (x + iy) + (x - iy) = 2x = 2\,\mathrm{Re}\,z$
(v) $z - \bar{z} = (x + iy) - (x - iy) = 2iy = 2i\,\mathrm{Im}\,z$. ■

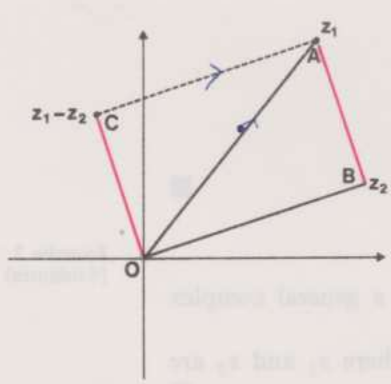
Solution 2 **Solution 2**

(i) The point representing $\bar{z}$ is the reflection in the real axis of the point representing z.



Note that if z has polar co-ordinates (r, θ), then $\bar{z}$ has polar co-ordinates $(r, -\theta)$.

(ii) If we represent z_1 and z_2 by geometric vectors, then the geometric vector representing $z_1 + z_2$ is obtained by adding the geometric vectors. Therefore, $z_1 - z_2$ is obtained by reversing the geometric vector representing z_2 and then adding it to the one representing z_1.



In the diagram $\underrightarrow{OA}$ represents z_1 and $\underrightarrow{OB}$ represents z_2. So that $\underrightarrow{AC}$ represents $-z_2$. Hence $z_1 - z_2$ is represented by $\underrightarrow{OC} = \underrightarrow{BA}$ and $|z_1 - z_2| = OC = BA$, i.e. $|z_1 - z_2|$ is the distance between the point representing z_1 and the point representing z_2. ■

Solution 3

Solution 3

We can use the formula

$$\frac{z_1}{z_2} = \frac{z_1\bar{z}_2}{|z_2|^2} \quad \text{or} \quad \frac{z_1\bar{z}_2}{z_2\bar{z}_2}.$$

Thus

(i) $\dfrac{1-i}{3+i} = \dfrac{(1-i)(3-i)}{3^2+1^2} = \dfrac{2-4i}{10} = \dfrac{1}{5} - \dfrac{2}{5}i$

(ii) $\dfrac{1}{1+i} + \dfrac{1+i}{i} = \dfrac{1-i}{1^2+1^2} + \dfrac{(-i)(1+i)}{1^2}$

$$= \frac{(1-i)+(2-2i)}{2} = \frac{3}{2} - \frac{3}{2}i$$

■

Discussion

We have introduced three functions:

$$\begin{aligned} &\text{Arg}: z \longmapsto \text{Arg}\, z && (z \in C, z \neq 0),\\ &\text{mod}: z \longmapsto |z| && (z \in C),\\ &\text{conj}: z \longmapsto \bar{z} && (z \in C), \end{aligned}$$

which turn up frequently in calculations with complex numbers. Their usefulness depends very much on a familiarity with their behaviour with respect to addition and multiplication. So in the following example and exercises we ask you to look at this.

Example 2

Example 2

We know that Arg is a many-one function and we have shown that

$$\text{Arg}\,(z_1 \otimes z_2) = \text{Arg}\, z_1 \oplus_{2\pi} \text{Arg}\, z_2 \qquad (z_1, z_2 \in C_1),$$

where C_1 is the set of non-zero complex numbers. (See Exercise 27.2.2.2.) So Arg is a morphism of $(C_1, \otimes)$ to $([0, 2\pi[, \oplus_{2\pi})$, or, put another way, Arg is compatible with $\otimes$ and the induced binary operation on the image set is $\oplus_{2\pi}$.

Is Arg compatible with addition? If so, what is the induced binary operation on the image set?

■

Solution of Example 2

The easiest way to deal with a question of compatibility when the result is not obvious is, first of all, to try some particular cases. Thus, in this exercise, we want to take z_1, z_2, z_3 and z_4 such that

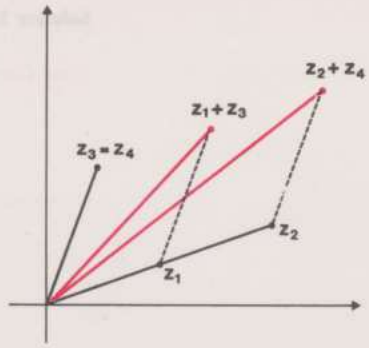
$$\text{Arg}\, z_1 = \text{Arg}\, z_2,$$
$$\text{Arg}\, z_3 = \text{Arg}\, z_4$$

and to look at

$$\text{Arg}\,(z_1 + z_3) \quad \text{and} \quad \text{Arg}\,(z_2 + z_4)$$

to see if they are equal or not. If not, then we have shown that Arg is not compatible with + by a counter-example (see *Unit 17, Logic II*). If they are equal, we must examine the question of compatibility further.

Now if $\text{Arg}\, z_1 = \text{Arg}\, z_2$, the geometric vectors representing z_1 and z_2 make the same angle with the real axis. The remainder of the argument refers to the following diagram. We have taken $z_3 = z_4$, since this simplifies the argument and still provides us with a counter-example.



It is clear from the diagram that

$$\operatorname{Arg}(z_1 + z_3) \neq \operatorname{Arg}(z_2 + z_4).$$

So Arg is not compatible with $+$, and there can be no operation $\square$ on $[0, 2\pi[$, for which Arg is a morphism of $(C_1, +)$ to $([0, 2\pi[, \square)$.

Exercise 4

Exercise 4
(4 minutes)

The modulus mapping is a many-one mapping of C to R_0^+. Is it compatible with addition or multiplication in C? If so, what are the corresponding binary operations on R_0^+? ■

Exercise 5

Exercise 5
(3 minutes)

The conjugate mapping is a one-one mapping of C to C. It is therefore compatible with both $+$ and $\otimes$. What are the induced binary operations on the codomain C? ■

Summary

Summary
**

In this section we have defined division by

$$z_1 \div z_2 = \frac{z_1}{z_2} = z_1 \otimes \frac{1}{z_2} = \frac{z_1 \bar{z}_2}{|z_2|^2} \qquad (z_2 \neq 0).$$

We have also proved the following results:

(i) $|z_1 z_2| = |z_1||z_2|$ (Exercise 4)
(ii) $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$ (Exercise 5)
(iii) $\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$ (Exercise 5)
(iv) $z\bar{z} = |z|^2$ (Exercise 1)
(v) $z + \bar{z} = 2 \operatorname{Re} z$ (Exercise 1)
(vi) $z - \bar{z} = 2i \operatorname{Im} z$ (Exercise 1).

The following results, although simple, are often useful:

(vii) $\overline{(\bar{z})} = z$
(viii) $(z = 0) \Leftrightarrow (|z| = 0)$
(ix) $|z| = |\bar{z}|$
(x) $|z| \geqslant \operatorname{Re} z$.

27.4.3 The Triangle Inequality

27.4.3

Main Text

We have seen that addition and the modulus mapping are not compatible, so we are unable to find an equation of the form

$$|z_1 + z_2| = |z_1| \square |z_2|.$$

We can, however, prove the triangle inequality:

Definition 1

$$|z_1 + z_2| \leqslant |z_1| + |z_2|.$$

We have already seen an equivalent result in *Unit 22, Linear Algebra I*, but it is interesting and instructive to prove it using complex algebra.

$$\begin{aligned}
|z_1 + z_2|^2 &= (z_1 + z_2)\overline{(z_1 + z_2)} && \text{(by (iv))}\\
&= (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) && \text{(by (ii))}\\
&= z_1\bar{z}_1 + z_2\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_2 && \text{(by distributivity)}\\
&= |z_1|^2 + z_2\bar{z}_1 + z_1\bar{z}_2 + |z_2|^2 && \text{(by (iv)).}
\end{aligned}$$

Notice that $\overline{z_2\bar{z}_1} = \bar{z}_2\bar{\bar{z}}_1$ (by (iii)) $= \bar{z}_2 z_1$ (by (vii)), i.e. $z_2\bar{z}_1$ is the conjugate of $z_1\bar{z}_2$. And if we add a complex number to its conjugate, then we get simply twice its real part (by (v)). Hence

$$|z_1 + z_2|^2 = |z_1|^2 + 2\,\mathrm{Re}\,(z_2\bar{z}_1) + |z_2|^2.$$

In the next exercise we ask you to prove that for any complex number z,

$$\mathrm{Re}\,(z) \leqslant |z|.$$

Applying this result to the complex number $z_2\bar{z}_1$, we have

$$\mathrm{Re}\,(z_2\bar{z}_1) \leqslant |z_2\bar{z}_1|$$

and hence

$$\begin{aligned}
|z_1 + z_2|^2 &\leqslant |z_1|^2 + 2|z_2\bar{z}_1| + |z_2|^2\\
&= |z_1|^2 + 2|z_2||\bar{z}_1| + |z_2|^2 && \text{(by (i))}\\
&= |z_1|^2 + 2|z_2||z_1| + |z_2|^2 && \text{(by (ix))}\\
&= (|z_1| + |z_2|)^2.
\end{aligned}$$

Since the modulus is positive or zero, we can now deduce that

$$|z_1 + z_2| \leqslant |z_1| + |z_2|.$$

Notice that this is an inequality between *real numbers.*

Exercise 1

Exercise 1
(4 minutes)

Prove that

(a) $\mathrm{Re}\,(z) \leqslant |z|$

(b) $|z_1 - z_2| \geqslant \big||z_1| - |z_2|\big|$. ■

Solution 27.4.2.4

Solution 27.4.2.4

We approach the solution to this problem in the same way as the previous example.

We start by choosing four numbers z_1, z_2, z_3 and z_4 such that

$$|z_1| = |z_2|$$

and

$$|z_3| = |z_4|.$$

We then look at

$$|z_1 + z_3| \quad \text{and} \quad |z_2 + z_4|$$

and

$$|z_1 \otimes z_3| \quad \text{and} \quad |z_2 \otimes z_4|.$$

We can either do this numerically, by choosing particular numbers, or by interpreting the problem geometrically, as in Example 2. Geometrically, if the moduli are equal, then the two numbers both lie on the circle with centre the origin and radius the common modulus. It is also worth remembering that $r = |z|$, and so it may be easier to work in polar co-ordinates.

By a suitable choice of z_1, z_2, z_3 and z_4, for example,

$$z_1 = 1, z_2 = 1,$$

$$z_3 = 1, z_4 = -1,$$

it can be shown that the modulus mapping is not compatible with addition. (See Example 1 in section 3.2.1, *Unit 3, Operations and Morphisms.*) On the other hand, in polar co-ordinates, we know that

$$\begin{array}{ccccc} (r_1, \theta_1) & \circ & (r_2, \theta_2) & = & (r_1 r_2, \theta_1 + \theta_2) \\ \downarrow & & \downarrow & & \downarrow \\ z_1 & & z_2 & & z_1 \otimes z_2 \end{array}$$

so that $|z_1 \otimes z_2| = r_1 r_2 = |z_1| \times |z_2|$.

Hence multiplication is compatible with the modulus mapping and the induced binary operation on R_0^+ is multiplication. ■

Solution 27.4.2.5

Solution 27.4.2.5

If $\quad z_1 = x_1 + iy_1 \quad \text{and} \quad z_2 = x_2 + iy_2,$

then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$$

and

$$\begin{aligned} \overline{z_1 + z_2} &= (x_1 + x_2) - i(y_1 + y_2) \\ &= (x_1 - iy_1) + (x_2 - iy_2) \\ &= \bar{z}_1 + \bar{z}_2. \end{aligned}$$

So the induced binary operation for addition is addition.

Similarly, it can be shown that

$$\overline{z_1 z_2} = \bar{z}_1 \bar{z}_2$$

so that the induced binary operation for multiplication is multiplication.

■

Solution 1 **Solution 1**

(a) If $z = x + iy$, then

$$|z|^2 = x^2 + y^2,$$

and since $y^2 \geqslant 0$

$$|z|^2 \geqslant x^2,$$

whence $|z| \geqslant x = \operatorname{Re}(z)$.

(b)
$$\begin{aligned}
|z_1 - z_2|^2 &= (z_1 - z_2)\overline{(z_1 - z_2)} && \text{(by (iv))}\\
&= (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) && \text{(by (ii))}\\
&= z_1\bar{z}_1 - z_2\bar{z}_1 - z_1\bar{z}_2 + z_2\bar{z}_2 && \text{(by distributivity)}\\
&= |z_1|^2 - 2\operatorname{Re}(z_2\bar{z}_1) + |z_2|^2 && \text{(by (v))}\\
&\geqslant |z_1|^2 - 2|z_2\bar{z}_1| + |z_2|^2 && \text{(by (x))}\\
&= |z_1|^2 - 2|z_1||z_2| + |z_2|^2 && \text{(by (i) and (ix))}\\
&= (|z_1| - |z_2|)^2.
\end{aligned}$$

Hence

$$|z_1 - z_2| \geqslant ||z_1| - |z_2||. \qquad ■$$

27.4.4 Products in Polar Form

27.4.4

Main Text

It is often a useful calculating device to convert a complex number $z = x + iy$ into its polar form* before multiplying, so that

$$x + iy = r(\cos\theta + i\sin\theta),$$

where θ is one value of $\arg z$.

This is particularly so when calculating high integer powers of z.

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then we know from our definition of multiplication that

$$z_1z_2 = r_1r_2(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

We can now see that if $z = r(\cos\theta + i\sin\theta)$, then

$$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$$

and, in general,

$$z^n = r^n(\cos n\theta + i\sin n\theta).$$

We shall prove the last result using the method of mathematical induction (see *Unit 17, Logic II*).

FIRST STEP

$$z^1 = r^1(\cos 1\theta + i\sin 1\theta).$$

SECOND STEP

Assume that

$$z^k = r^k(\cos k\theta + i\sin k\theta) \qquad \text{for some } k \in Z^+.$$

* Some authors abbreviate $\cos\theta + i\sin\theta$ to $\operatorname{cis}\theta$.

Then

$$z^{k+1} = r^k(\cos k\theta + i\sin k\theta)r(\cos\theta + i\sin\theta)$$
$$= r^{k+1}((\cos k\theta\cos\theta - \sin k\theta\sin\theta)$$
$$+ i(\sin k\theta\cos\theta + \cos k\theta\sin\theta)),$$

i.e. $$z^{k+1} = r^{k+1}(\cos(k+1)\theta + i\sin(k+1)\theta).$$ (See RB10)

Since the conjecture is TRUE for $n = 1$, and also for $n = k + 1$ whenever it is TRUE for $n = k$ ($k \in Z^+$), it follows that it is TRUE for all $n \in Z^+$.

Notice also that $z^n = r^n(\cos\theta + i\sin\theta)^n$ so that

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

for any positive integer n. This result is a special case of a theorem known as De Moivre's Theorem.

Example 1 **Example 1**

To evaluate $(1 + i)^{10}$ we first write $1 + i$ in its polar form

$$\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right).$$

Then

$$(1 + i)^{10} = (\sqrt{2})^{10}\left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right)$$
$$= 32(0 + i) = 32i$$

The method of solution is essentially the same as before but De Moivre's Theorem enables us to condense the work. ■

Exercise 1 **Exercise 1** (4 minutes)

Use the fact that

$$(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$$

to prove that

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta.$$ ■

Exercise 2 **Exercise 2** (3 minutes)

If $z = \frac{1}{4}(1 + i\sqrt{3})$,

calculate $|z^n|$ for any positive integer n. ■

27.4.5 Sets of Points in the Complex Plane

27.4.5

Main Text

Often it is useful to specify a particular subset of the complex plane, and sometimes this can be done concisely in terms of argument, conjugate and modulus.

Example 1

Example 1

Indicate the set $\{z : z = \bar{z}\}$ on an Argand diagram.

If $z = x + iy$ and $z = \bar{z}$, then

$$x + iy = x - iy$$

Equating imaginary parts, we get $y = -y$, so that $y = 0$.

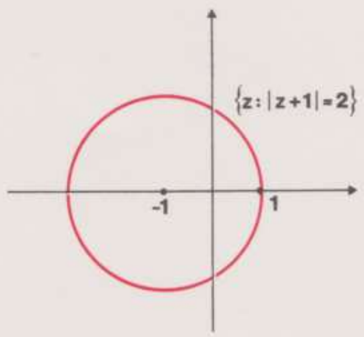
In other words, the set coincides with the real axis. ■

Example 2

Example 2

Indicate the set $\{z : |z + 1| = 2\}$ on an Argand diagram.

If $|z + 1| = 2$ then "the distance of z from -1 is 2", so that z lies on a circle with centre $(-1, 0)$ and radius 2.



■

Exercise 1

Exercise 1
(3 minutes)

Indicate the following sets on an Argand diagram.

(i) $\{z : |z - 1| \leqslant 2\}$

(ii) $\left\{z : \operatorname{Arg} z = \frac{\pi}{4}\right\}$

(iii) $\{z : z + 2\bar{z} = 1\}$

(iv) $\left\{z : \operatorname{Arg}(z - 1) = \frac{\pi}{4}\right\}$

(v) $\{z : 0 \leqslant \operatorname{Arg} z \leqslant \pi\}$

(vi) $\left\{z : 0 \leqslant \operatorname{Arg} z \leqslant \frac{\pi}{2}, |z| \leqslant 1\right\}$

(vii) $\{z : |z - 1| = |z + 1|\}$

(viii) $\{z : |z - 1| \leqslant |z + 1|\}$ ■

Solution 27.4.4.1 **Solution 27.4.4.1**

With $n = 3$, we have

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta.$$

On the other hand, multiplying out, we have

$$(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

Equating the real parts of the two right-hand sides gives the required result. ■

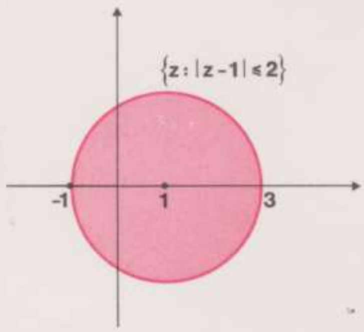
Solution 27.4.4.2 **Solution 27.4.4.2**

Did you get stuck? If so, it's probably because you tried to work out z^n before taking the modulus.

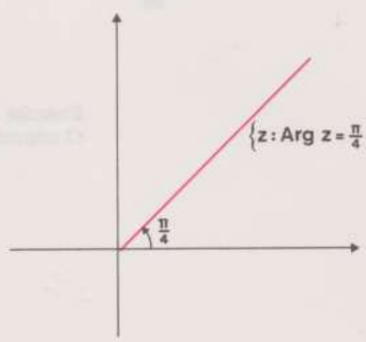
Use the fact that $|z^n| = |z|^n$. The answer is $(\frac{1}{2})^n$. ■

Solution 1 **Solution 1**

(i) "The distance of z from 1 is less than or equal to 2".



(ii)

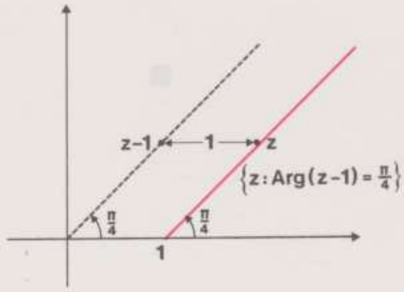


(iii) If $z + 2\bar{z} = 1$ and $z = x + iy$, then

$$3x - iy = 1,$$

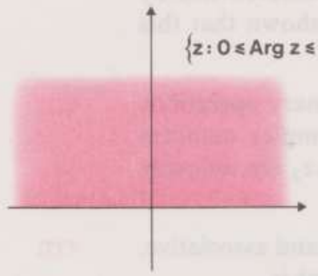
so that $3x = 1$ and $y = 0$. The answer is a diagram showing the single point $(\frac{1}{3}, 0)$.

(iv)



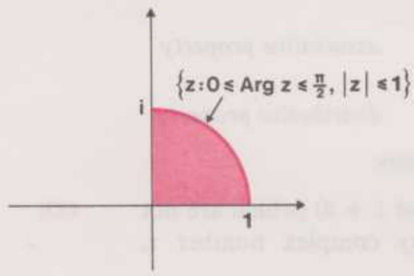
The solution is a "half line" at an angle $\frac{\pi}{4}$ to the real axis through the point $z = 1$.

(v)

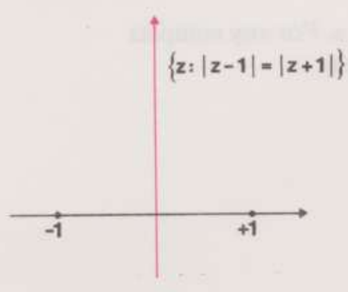


The solution is the upper half-plane including the real axis.

(vi)

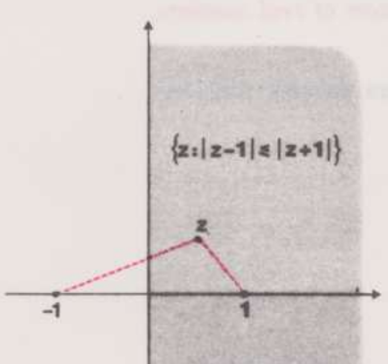


(vii)



"The distance of z from 1 equals the distance of z from -1", so the set coincides with the imaginary axis.

(viii)



"The distance of z from 1 is less than, or equal to, the distance of z from -1". The set is the right half-plane including the imaginary axis. ■

27.5 CONCLUSION

27.5

Summary

* * *

We set ourselves the task of extending the real number system in order to include the solution of the equation $z^2 = -1$. We achieved this by defining the system of complex numbers, and we have shown that this system has the following properties:

C(1) The set of complex numbers is closed under the binary operations of addition and multiplication. That is, for any complex numbers z_1 and z_2, both the sum $z_1 + z_2$ and the product z_1z_2 are uniquely defined and are themselves complex numbers. C(1)

C(2) Addition and multiplication are both commutative and associative, and multiplication is distributive over addition. That is, C(2)

$$\left.\begin{aligned} z_1 + z_2 &= z_2 + z_1 \\ z_1z_2 &= z_2z_1 \end{aligned}\right\} \quad \textit{commutative property}$$

$$\left.\begin{aligned} z_1 + (z_2 + z_3) &= (z_1 + z_2) + z_3 \\ z_1(z_2z_3) &= (z_1z_2)z_3 \end{aligned}\right\} \quad \textit{associative property}$$

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3 \qquad \textit{distributive property}$$

where z_1, z_2 and z_3 are any complex numbers.

C(3) There are two complex numbers, $0 + i0$ and $1 + i0$ (which are not equal), with the properties that, for any complex number z, $z + (0 + i0) = z$ and $z(1 + i0) = z$. C(3)

C(4) For any complex numbers α and β, the equation C(4)

$$\alpha + z = \beta$$

has a unique complex solution, written $z = \beta - \alpha$. For any complex numbers α and β, where $\alpha \neq 0 + i0$, the equation

$$\alpha z = \beta$$

has a unique complex solution, written

$$\frac{\beta}{\alpha}.$$

You should compare these properties carefully with the properties Re(1), Re(2), Re(3) and Re(4) of the real number system, given in section 6.1.1 of *Unit 6, Inequalities*. The properties C(1), C(2), C(3) and C(4) correspond to the properties Re(1), Re(2), Re(3) and Re(4) respectively of the real number system. That is, *we have extended the system of real numbers, while preserving these important properties.*

Our extension to the system of complex numbers ensures that any quadratic equation has two roots. Suppose that

$$w^2 + bw + c = 0,$$

then

$$w = \frac{-b \pm \sqrt{b^2 - 4c}}{2}.$$

If $b^2 \geqslant 4c$, then there are two real roots (which are the same if $b^2 = 4c$). If $b^2 < 4c$, then, according to Pacioli, the equation was "impossible" (i.e. had an empty solution set), and we would certainly agree with him if the equation applies only to real numbers. If, however, we are asked to find complex numbers satisfying

$$w^2 + bw + c = 0,$$

then we can *always* find two such numbers. If $b^2 < 4c$, it is easy to verify that

$$w = \frac{-b \pm i\sqrt{4c - b^2}}{2}$$

are suitable numbers.

It is possible to prove that every polynomial of degree n has exactly n complex roots (if we make allowance for repeated roots). For example, the equation

$$z^4 = -1$$

has four roots

$$\frac{1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}, \frac{-1+i}{\sqrt{2}}, \frac{1-i}{\sqrt{2}}.$$

You can easily verify that these numbers are indeed roots of the equation, but you may well ask how we were able to find them. That is something which we shall tackle in the next unit on complex numbers.

In the next unit we shall also discuss complex functions whose domains and codomains are subsets of C. Unfortunately we shall not have time to show you some of the delights of complex analysis, which is the subject which develops when the underlying ideas of calculus are applied to complex functions.

Acknowledgements

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Text

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